Identities from the Gould notebooks, Egorychev method

Marko R. Riedel

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We apply Egorychev method to some identities from the Notebooks of H.W.Gould: [Gou11].

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1 Identity 2.13 Vol 4

We seek the following identity:

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\frac{1}{n^3-k^2} = \frac{1}{2n^3}\binom{2n}{n} + \frac{(-1)^n}{2n^2\sqrt{n}(\sqrt{n}-1)\binom{n+n\sqrt{n}}{2n}}.$$

We will start by evaluating

$$S_n(x) = \sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\frac{1}{x^2-k^2}.$$

Note that

$$x^2\binom{x-1}{n}\binom{x+n}{n}S_n(x)$$

is a polynomial of degree $2n$ in x. Therefore if we can find a closed form for this product and it is also a polynomial it suffices to show equality for x a positive integer not in $[0,n]$ and we have equality for complex x.

This is

$$\frac{1}{2x}\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\left[\frac{1}{x+k}+\frac{1}{x-k}\right]$$

First piece

$$\frac{(-1)^n}{2x}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\frac{1}{x+n-k}$$

$$= [v^{n+x}]\log\frac{1}{1-v}\frac{(-1)^n}{2x}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}v^k$$

$$= [v^{n+x}]\log\frac{1}{1-v}[z^n]\frac{1}{1-z}\frac{(-1)^n}{2x}\sum_{k\ge 0}(-1)^k\binom{2n}{k}v^k z^k$$

$$= [v^{n+x}]\log\frac{1}{1-v}[z^n]\frac{1}{1-z}\frac{(-1)^n}{2x}(1-vz)^{2n}.$$

The contribution from z is

$$\operatorname*{res}_z \frac{1}{z^{n+1}}\frac{1}{1-z}(1-vz)^{2n}.$$

We evaluate using minus the residue at $z=1$ and at infinity. We find for the former,

$$[v^{n+x}]\log\frac{1}{1-v}(1-v)^{2n}$$

and for the latter

$$\begin{aligned}\operatorname*{res}_z \frac{1}{z^2} z^{n+1} \frac{1}{1-1/z} (1-v/z)^{2n} &= - \operatorname*{res}_z \frac{1}{z^n} \frac{1}{1-z} (z-v)^{2n} \\ &= - \operatorname*{res}_z \frac{1}{z^{n+1}} \frac{z}{1-z} (z-v)^{2n}\end{aligned}$$

Second piece

$$\frac{1}{2x} \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \frac{1}{x-k}$$

$$= [v^x] \log \frac{1}{1-v} \frac{1}{2x} \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} v^k$$

$$= [v^x] \log \frac{1}{1-v} \frac{(-1)^n}{2x} \sum_{k=0}^{n} (-1)^k \binom{2n}{k} v^{n-k}$$

$$= [v^{x-n}] \log \frac{1}{1-v} [z^n] \frac{1}{1-z} \frac{(-1)^n}{2x} \sum_{k\ge 0} (-1)^k \binom{2n}{k} v^{-k} z^k$$

$$= [v^{x-n}] \log \frac{1}{1-v} [z^n] \frac{1}{1-z} \frac{(-1)^n}{2x} (1-z/v)^{2n}$$

$$= [v^{n+x}] \log \frac{1}{1-v} [z^n] \frac{1}{1-z} \frac{(-1)^n}{2x} (v-z)^{2n}.$$

Next phase

We see that on combining the two residues in z we are left with

$$\operatorname*{res}_z \frac{1}{z^{n+1}} (z-v)^{2n} = \binom{2n}{n} (-1)^n v^n.$$

This gives with the extractor in v collecting everything

$$\binom{2n}{n} \frac{1}{2x^2}.$$

This means we can conclude if we find

$$[v^{n+x}] \log \frac{1}{1-v} (1-v)^{2n}$$

$$= [v^{n+x}] \log \frac{1}{1-v} \sum_{q=0}^{2n} \binom{2n}{q} (-1)^q v^q = \sum_{q=0}^{2n} \binom{2n}{q} \frac{(-1)^q}{n+x-q}.$$

There is no problem with the division since we said that $x \ge n+1$. Now introduce

$$F(z) = (2n)! \frac{1}{n+x-z} \prod_{r=0}^{2n} \frac{1}{z-r}.$$

This has the property that with $0 \le q \le 2n$

$$\operatorname*{res}_{z=q} F(z) = (2n)! \frac{1}{n+x-q} \prod_{r=0}^{q-1} \frac{1}{q-r} \prod_{r=q+1}^{2n} \frac{1}{q-r}$$

$$= (2n)! \frac{1}{n+x-q} \frac{1}{q!} (-1)^{2n-q} \frac{1}{(2n-q)!} = \binom{2n}{q} (-1)^q \frac{1}{n+x-q}.$$

Hence we may evaluate the remaining sum using minus the residue at $z = n+x$, getting

$$(2n)! \prod_{r=0}^{2n} \frac{1}{n+x-r} = \binom{n+x}{2n}^{-1} \frac{1}{x-n}.$$

We have shown that

$$\boxed{\sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \frac{1}{x^2-k^2} = \frac{1}{2x^2}\binom{2n}{n} + (-1)^n \binom{n+x}{2n}^{-1} \frac{1}{2x(x-n)}.}$$

We see that upon multiplication by $x^2 \binom{x-1}{n}\binom{x+n}{n}$ we obtain a polynomial of degree $2n$ as claimed and hence we have established the identity for all x such that $x \notin [-n, n]$. The value $x = n^{3/2}$ fits this requirement and on substituting we get the claimed identity that we sought to prove.

2 Identity 3.31 Vol 4

We seek the following identity:

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \sum_{j=1}^{2k} \frac{1}{j} = \frac{1}{2n} + \frac{2^{2n-1}}{n} \binom{2n}{n}^{-1}.$$

We have from first principles

$$(-1)^{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} [v^{2n-2k}] \frac{1}{1-v} \log \frac{1}{1-v}$$

$$= (-1)^{n-1} [v^{2n}] \frac{1}{1-v} \log \frac{1}{1-v} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} v^{2k}.$$

Here we are missing the $k = n$ term. It results that it is zero, so we may add it in:

$$(-1)^{n-1}[v^{2n}]\frac{1}{1-v}\log\frac{1}{1-v}(-1)^n v^{2n} = 0$$

and we get

$$(-1)^{n-1}[v^{2n}]\frac{1}{1-v}\log\frac{1}{1-v}(1-v^2)^n$$
$$= (-1)^{n-1}[v^{2n}]\log\frac{1}{1-v}(1-v)^{n-1}(1+v)^n.$$

Expand the rightmost power

$$[v^{2n}]\log\frac{1}{1-v}(v-1)^{n-1}\sum_{q=0}^{n}\binom{n}{q}2^{n-q}(v-1)^q$$
$$= [v^{2n}]\log\frac{1}{1-v}\sum_{q=0}^{n}\binom{n}{q}2^{n-q}(v-1)^{n-1+q}.$$

Recall the inverse binomial coefficient identity

$$\binom{n-1}{k-1}^{-1} = n[v^n]\log\frac{1}{1-v}(v-1)^{n-k}.$$

Apply to our sum,

$$\frac{1}{2n}\sum_{q=0}^{n}\binom{n}{q}2^{n-q}\binom{2n-1}{n-q}^{-1}.$$

Now observe that

$$\binom{n}{q}\binom{2n-1}{n-q}^{-1} = \frac{n!\times(n-1+q)!}{q!\times(2n-1)!} = \binom{2n-1}{n}^{-1}\binom{n-1+q}{q}.$$

We are left with

$$\frac{1}{2n}\binom{2n-1}{n}^{-1}\sum_{q=0}^{n}\binom{n-1+q}{q}2^{n-q}$$
$$= \frac{2^n}{2n}\binom{2n-1}{n}^{-1}[z^n]\frac{1}{1-z}\frac{1}{(1-z/2)^n}.$$

Here the contribution from z is

$$\operatorname*{res}_{z}\frac{1}{z^{n+1}}\frac{1}{1-z}\frac{1}{(1-z/2)^n}.$$

Now put $z = 1-\sqrt{1-2w}$ so that $z(1-z/2) = w$ and $dz = \frac{1}{\sqrt{1-2w}}\, dw$ to get

$$\operatorname*{res}_{w} \frac{1}{1-\sqrt{1-2w}} \frac{1}{w^n} \frac{1}{\sqrt{1-2w}} \frac{1}{\sqrt{1-2w}}$$
$$= \operatorname*{res}_{w} \frac{1+\sqrt{1-2w}}{1-(1-2w)} \frac{1}{w^n} \frac{1}{1-2w}$$
$$= \frac{1}{2} \operatorname*{res}_{w} (1+\sqrt{1-2w}) \frac{1}{w^{n+1}} \frac{1}{1-2w}.$$

Actually computing the residue we get

$$2^{n-1} + \frac{1}{2} \operatorname*{res}_{w} \frac{1}{w^{n+1}} \frac{1}{\sqrt{1-2w}} = 2^{n-1} + \frac{1}{2} \binom{2n}{n} \frac{1}{2^n}.$$

Collecting everything that we have

$$\frac{2^n}{2n} \binom{2n-1}{n}^{-1} \left[2^{n-1} + \frac{1}{2} \binom{2n}{n} \frac{1}{2^n} \right]$$
$$= \frac{2^n}{2n} \binom{2n}{n}^{-1} \frac{2n}{n} \left[2^{n-1} + \frac{1}{2} \binom{2n}{n} \frac{1}{2^n} \right]$$
$$= 2^{2n-1} \frac{1}{n} \binom{2n}{n}^{-1} + \frac{1}{2n}.$$

This is the claim.

The inverse binomial coefficient identity is from the paper "Inverse binomial coefficients in Egorychev method" by M. Riedel, M. Scheuer and H. Mahmoud, [Mar25].

3 Identity 10.38 Vol 4

We seek the following identity:

$$S_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n \left(1 + \frac{1}{k}\right)^k = (-1)^n \lfloor e \times n! \rfloor$$

where we have the usual floor function.

We expand the approximant to Euler's number and get

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n \sum_{q=0}^{k} \binom{k}{q} \frac{1}{k^q}.$$

Now the term for $q = 0$ gives

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n = n! [w^n] \sum_{k=0}^{n} (-1)^k \binom{n}{k} \exp(kw).$$

Here we have lowered to $k = 0$ because with the extractor there is no contribution from that constant term. Continuing,

$$n![w^n](1 - \exp(w))^n = (-1)^n n!.$$

We may now switch sums to get

$$\sum_{q=1}^{n} \sum_{k=q}^{n} \binom{n}{k} \binom{k}{q} (-1)^k k^{n-q}.$$

We write

$$\binom{n}{k}\binom{k}{q} = \frac{n!}{(n-k)! \times q! \times (k-q)!} = \binom{n}{q}\binom{n-q}{k-q}$$

and obtain

$$\sum_{q=1}^{n} \binom{n}{q} \sum_{k=q}^{n} \binom{n-q}{k-q} (-1)^k k^{n-q}$$

$$= \sum_{q=1}^{n} \binom{n}{q} (-1)^q \sum_{k=0}^{n-q} \binom{n-q}{k} (-1)^k (k+q)^{n-q}$$

$$= \sum_{q=1}^{n} \binom{n}{q} (-1)^q (n-q)! [w^{n-q}] \sum_{k=0}^{n-q} \binom{n-q}{k} (-1)^k \exp(w(k+q))$$

$$= \sum_{q=1}^{n} \binom{n}{q} (-1)^q (n-q)! [w^{n-q}] \exp(wq)(1 - \exp(w))^{n-q}$$

$$= n![w^n] \sum_{q=0}^{n} \frac{1}{q!} (-1)^q w^q \exp(wq)(1 - \exp(w))^{n-q}.$$

Here we have lowered q back to zero to account for the correction term we computed.

We would like to be able to extend the sum to infinity but when $q > n$ say $q = n + p$ we get series A which is

$$w^{n+p} \exp(w(n+p)) \frac{1}{(1 - \exp(w))^p}$$

$$= \exp(w(n+p)) w^n \frac{1}{((1 - \exp(w))/w)^p}$$

$$= \exp(w(n+p)) w^n \frac{1}{(-1 - w/2 - w^2/6 - \cdots)^p}.$$

These terms all contribute to $[w^n]$ with $(-1)^{n-q}$, hence we may extend the sum to infinity but we must subtract a correction term which is

$$-(-1)^n n! \sum_{q\ge n+1} \frac{1}{q!}.$$

The remaining terms from the series A do not contribute to $[w^n]$ and we may raise to infinity to get

$$n![w^n](1-\exp(w))^n \exp\left[-\frac{w\exp(w)}{1-\exp(w)}\right].$$

We observe that the first term after the extractor has expansion $(-1)^n w^n + \cdots$ hence only the constant term from the bracketed term can possibly contribute, which gives

$$(-1)^n n! [w^0] \exp\left[-\frac{\exp(w)}{(1-\exp(w))/w}\right]$$
$$= (-1)^n n! [w^0] \exp\left[-\frac{\exp(w)}{-1-w/2-w^2/6-\cdots}\right]$$
$$= (-1)^n n! \exp(1).$$

We thus obtain

$$(-1)^n n! \left[\exp(1) - \sum_{q\ge n+1} \frac{1}{q!}\right].$$

This means that the multiple of $\exp(1)$ overestimates. When n is even the term is above the sum, and when n is odd, below. We have for the error term that

$$\sum_{q\ge n+1} \frac{n!}{q!} < \sum_{q\ge n+1} \frac{1}{(n+1)^{q-n}} = \sum_{q\ge 0} \frac{1}{(n+1)^{q+1}} = \frac{1}{n+1}\frac{1}{1-1/(n+1)} = \frac{1}{n}.$$

Observe that by construction S_n is an integer. The estimate from above / below is off by at most $1/n < 1$ from S_n hence the floor of the estimate is equal to S_n.

4 Identity 12.9 Vol 4

We seek the following identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2k+1} \binom{n+k+1}{k}^{-1} = \frac{4^{2n}}{n+1} \binom{2n+1}{n}^{-2}.$$

On multiplying both sides by $\binom{2n+1}{n}$ we get inside the sum

$$\binom{n+k+1}{k}^{-1}\binom{2n+1}{n+1} = \frac{(2n+1)! \times k!}{(n+k+1)! \times n!} = \binom{2n+1}{n+k+1}\binom{n}{k}^{-1}.$$

This leaves for the sum

$$S_n = \sum_{k=0}^{n}(-1)^k \frac{1}{2k+1}\binom{2n+1}{n-k} = (-1)^n \sum_{k=0}^{n}(-1)^k \frac{1}{2n-2k+1}\binom{2n+1}{k}$$

$$= (-1)^n \sum_{k=0}^{2n+1}(-1)^k \frac{1}{2n-2k+1}\binom{2n+1}{k} - (-1)^n \sum_{k=n+1}^{2n+1}(-1)^k \frac{1}{2n-2k+1}\binom{2n+1}{k}.$$

Working with the second sum,

$$-(-1)^n \sum_{k=0}^{n}(-1)^{2n+1-k} \frac{1}{2n-2(2n+1-k)+1}\binom{2n+1}{k}$$

$$= (-1)^n \sum_{k=0}^{n}(-1)^k \frac{1}{2k-(2n+1)}\binom{2n+1}{k} = -S_n.$$

We have shown that

$$S_n = \frac{1}{2}(-1)^n \sum_{k=0}^{2n+1}(-1)^k \frac{1}{2n-2k+1}\binom{2n+1}{k}.$$

The evaluation of this sum is standard procedure. We introduce

$$F(z) = \frac{1}{2}(-1)^{n+1}(2n+1)! \frac{1}{2n+1-2z} \prod_{r=0}^{2n+1} \frac{1}{z-r}$$

$$= \frac{1}{4}(-1)^{n+1}(2n+1)! \frac{1}{n+1/2-z} \prod_{r=0}^{2n+1} \frac{1}{z-r}$$

which has the property that with $0 \le k \le 2n+1$

$$\operatorname*{res}_{z=k} F(z) = \frac{1}{2}(-1)^{n+1}(2n+1)! \frac{1}{2n+1-2k} \prod_{r=0}^{k-1} \frac{1}{k-r} \prod_{r=k+1}^{2n+1} \frac{1}{k-r}$$

$$= \frac{1}{2}(-1)^{n+1}(2n+1)! \frac{1}{2n+1-2k} \frac{1}{k!}(-1)^{2n-k+1} \frac{1}{(2n+1-k)!}$$

$$= \frac{1}{2}(-1)^n \frac{1}{2n+1-2k}(-1)^k \binom{2n+1}{k}$$

so that these residues add up to precisely our sum. With the residue at infinity being zero we may evaluate with minus the residue at $z = n + 1/2$ to get

$$\frac{1}{4}(2n+1)!(-1)^{n+1} \prod_{r=0}^{2n+1} \frac{1}{n+1/2-r}$$
$$= 2^{2n}(2n+1)!(-1)^{n+1} \prod_{r=0}^{2n+1} \frac{1}{2n+1-2r}$$
$$= 2^{2n}(2n+1)!(-1)^{n+1} \prod_{r=0}^{n} \frac{1}{2n+1-2r} \prod_{r=n+1}^{2n+1} \frac{1}{2n+1-2r}$$
$$= 2^{2n}(2n+1)!(-1)^{n+1} \frac{1}{(2n+1)!!} \prod_{r=0}^{n} \frac{1}{-2r-1}$$
$$= 2^{2n}(2n+1)! \frac{1}{(2n+1)!!} \frac{1}{(2n+1)!!}$$
$$= 2^{2n} n! 2^n \frac{n! 2^n}{(2n+1)!} = 4^{2n} \frac{1}{n+1} {\binom{2n+1}{n}}^{-1}.$$

This is the claim and we may conclude.

5 Identity 4.6 Vol 5

We seek the following identity where $u \neq v$ and $v \neq 0$ are complex numbers:

$$v^n \sum_{k=0}^{n} \binom{\frac{u}{v}k}{k} \binom{n-\frac{u}{v}k}{n-k} = \frac{u^{n+1}-v^{n+1}}{u-v}.$$

We re-write by setting $u = vx$ and move the factor:

$$\sum_{k=0}^{n} \binom{xk}{k} \binom{n-xk}{n-k} = \frac{1}{v^n} \frac{v^{n+1}(x^{n+1}-1)}{v(x-1)} = \frac{x^{n+1}-1}{x-1}.$$

With the conditions on u and v we see that $x = 1$ is excluded. As we have a polynomial in x of degree n on both sides it is sufficient to prove it for $x \geq 2$ a positive integer and we then have it for all x.

We first have the extractor integral

$$\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{1}{z^{n+1}} (1+z)^n \sum_{k\geq 0} \binom{xk}{k} \frac{z^k}{(1+z)^{xk}} \; dz.$$

Here we have extended to infinity because the residue vanishes when $k \geq n+1$. Continuing,

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+1}}(1+z)^n \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w}\sum_{k\ge 0} \frac{1}{w^k}(1+w)^{xk} \frac{z^k}{(1+z)^{xk}} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+1}}(1+z)^n \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w}\frac{1}{1-(1+w)^x z/(1+z)^x/w} \; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+1}}(1+z)^{n+x} \frac{1}{2\pi i}\int_{|w|=\gamma} \frac{1}{w(1+z)^x-(1+w)^x z} \; dw\; dz.$$

Here we need convergence of the geometric series and must instantiate our contours. We need $\varepsilon, \gamma < 1$ from the integrals for the binomial coefficients. We require $|z/(1+z)^x| < |w/(1+w)^x|$. With $\varepsilon = 1/Q^2$ where Q is large we get the bound $|z/(1+z)^x| < 1/Q^2/(1-1/Q^2)^x = Q^{2x-2}/(Q^2-1)^x$. If we then set $\gamma = 1/Q$ we get the lower bound $|w/(1+w)^x| > 1/Q/(1/Q+1)^x = Q^{x-1}/(Q+1)^x$. Hence we need only verify that $1 > Q^{x-1}/(Q-1)^x$ or $1 > (1/Q)(Q/(Q-1))^x$. We now choose $Q = Q(x) = x+1$ and get $1/(x+1)(1+1/x)^x < 1/(x+1)\exp(1) < 1$. We are justified in summing the geometric series.

Note that with $\gamma > \varepsilon$ the pole at $w = z$ is inside the contour and we get the residue

$$\left.\frac{1}{(1+z)^x - x(1+w)^{x-1}z}\right|_{w=z} = \frac{1}{(1+z)^x - x(1+z)^{x-1}z}$$

$$= \frac{1}{(1+z)^{x-1}}\frac{1}{1+z-xz} = \frac{1}{(1+z)^{x-1}}\frac{1}{1+z(1-x)}.$$

This gives for z

$$\frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{z^{n+1}}(1+z)^{n+1}\frac{1}{1+z(1-x)} \; dz.$$

Now put $z/(1+z) = s$ so that $z = s/(1-s)$ and $dz = 1/(1-s)^2\; ds$ to get

$$\frac{1}{2\pi i}\int_{|s|=\varepsilon} \frac{1}{s^{n+1}}\frac{1}{1+s(1-x)/(1-s)}\frac{1}{(1-s)^2} \; ds$$

$$= \frac{1}{2\pi i}\int_{|s|=\varepsilon} \frac{1}{s^{n+1}}\frac{1}{1-sx}\frac{1}{1-s} \; ds.$$

The image of $|z| = \varepsilon$ under this substitution is $s = z - z^2 + z^3 \mp \cdots$ which makes one turn and is contained in an annulus with radii $\varepsilon \pm \varepsilon^2/(1-\varepsilon)$. Hence we may deform to $|s| = \varepsilon$ as this does not affect the two outer poles at $s = 1$ and $s = 1/x$ which are outside the annulus. Doing the extraction we find

$$\sum_{q=0}^{n} x^q = \frac{x^{n+1}-1}{x-1}$$

as claimed. We still need to verify that there are no other poles inside the contour. We apply Rouché's theorem with $f(z) = w(1+z)^x$ and $g(z) = (1+w)^x z$ to show that there is in fact just one pole, at $w = z$. The contour for the application of the theorem is our $|w| = \gamma$. We see that $f(z)$ dominates on this contour because it is lower bounded by $(1/Q)(1 - 1/Q^2)^x$ and $g(z)$ is upper bounded by $(1/Q^2)(1 + 1/Q)^x$. We then need to check that $|f(z)| \ge |g(z)|$ or $Q(1 - 1/Q)^x > 1$. We have with everything being positive the logarithm

$$\begin{aligned}
&\log(1+x) + x\log(1 - 1/(1+x)) \\
&= \log(1+x) - x\log(1/(1 - 1/(1+x))) \\
&= \log(1+x) - x((1+x) + (1+x)^2/2 + (1+x)^3/3 + \cdots) \\
&> \log(1+x) - x\frac{(1+x)}{(-x)} = \log(1+x) + 1 + x > 0.
\end{aligned}$$

Hence we may conclude that $f(z)+g(z)$ has the same number of zeros inside the contour as $f(z) = w(1+z)^x$ which is to say, just one, at $w = z$. This also shows the pole in the geometric term must be simple.

Proof by residue operators

We start with the following usual representation,

$$\binom{xk}{k} = \operatorname*{res}_z \frac{1}{z^{k+1}}(1+z)^{xk}.$$

Now we introduce the inverse $Q(w)$ of the function $w = z/(1+z)^x$ so that $Q(z/(1+z)^x) = z$ or $z/(1+z)^x = Q^{-1}(z)$. We make the substitution $w = z/(1+z)^x$ which is valid as a formal power series so that $z = Q(w)$ and $dz = Q'(w)\, dw$ to get

$$\binom{xk}{k} = \operatorname*{res}_w \frac{1}{Q(w)}\frac{1}{w^k}Q'(w).$$

We find for our sum,

$$\begin{aligned}
&[s^n](1+s)^n \operatorname*{res}_w \frac{1}{Q(w)}Q'(w)\sum_{k\ge 0}\frac{1}{w^k}\frac{s^k}{(1+s)^{xk}} \\
&= [s^n](1+s)^n \sum_{k\ge 0}\frac{s^k}{(1+s)^{xk}}[w^k]\frac{w}{Q(w)}Q'(w) \\
&= [s^n](1+s)^n \frac{s}{(1+s)^x}\frac{Q'(s/(1+s)^x)}{Q(s/(1+s)^x)}.
\end{aligned}$$

Note that the pole of $Q(w)$ at zero is simple so on multiplying the inverse by w we get a proper formal power series. We get from the definition of the derivative that

$$Q'(z/(1+z)^x)(z/(1+z)^x)' = 1$$

or

$$Q'(z/(1+z)^x)(z/(1+z)^x)(1/z - x/(1+z)) = 1.$$

Substitute

$$[s^n](1+s)^n \frac{1}{1/s - x/(1+s)} \frac{1}{s} = [s^n](1+s)^n \frac{s(1+s)}{1+s-sx} \frac{1}{s}$$
$$= [s^n](1+s)^{n+1} \frac{1}{1+s(1-x)}$$
$$= \sum_{q=0}^{n} \binom{n+1}{n-q} (x-1)^q = \sum_{q=0}^{n} \binom{n+1}{q+1} (x-1)^q$$
$$= \frac{1}{x-1} \sum_{q=1}^{n+1} \binom{n+1}{q} (x-1)^q = \frac{1}{x-1} \left(x^{n+1} - 1\right).$$

6 Identity 9.20 Vol 5

We seek the following identity:

$$\sum_{k=0}^{n} \binom{n}{k}^4 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{2n-j}{n} \sum_{k=0}^{j} (-1)^k \binom{n}{k}^2 \binom{j}{k}.$$

We start with the inner sum and get

$$\sum_{k=0}^{j} (-1)^k \binom{n}{k}^2 \binom{j}{k}$$
$$= [z^n](1+z)^n [w^n](1+w)^n \sum_{k=0}^{j} (-1)^k z^k w^k \binom{j}{k}$$
$$= [z^n](1+z)^n [w^n](1+w)^n (1-wz)^j.$$

Substituting into the outer sum we find

$$[z^n](1+z)^n [w^n](1+w)^n [v^n](1+v)^{2n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (1-wz)^j \frac{1}{(1+v)^j}$$
$$= [z^n](1+z)^n [w^n](1+w)^n [v^n](1+v)^{2n} \left[1 - \frac{1-wz}{1+v}\right]^n$$
$$= [z^n](1+z)^n [w^n](1+w)^n [v^n](1+v)^n [1+v-(1-wz)]^n$$
$$= [z^n](1+z)^n [w^n](1+w)^n [v^n](1+v)^n [v+wz]^n.$$

Now expanding the powered term we have

$$[z^n](1+z)^n[w^n](1+w)^n[v^n](1+v)^n \sum_{k=0}^{n} \binom{n}{k} v^k (wz)^{n-k}$$

$$= [z^n](1+z)^n[w^n](1+w)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}(wz)^{n-k}$$

$$= [z^n](1+z)^n[w^n](1+w)^n \sum_{k=0}^{n} \binom{n}{k}^2 (wz)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 [z^k](1+z)^n[w^k](1+w)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n}{k}\binom{n}{k}.$$

This is the claim.

7 Identity 10.39 Vol 5

We seek the following identity:

$$\sum_{k=0}^{3n}(-1)^k \binom{3n}{k}\binom{5n+k}{4n}\frac{1}{2^k} = (-1)^{3n/2}\frac{1}{2^{3n}}\binom{4n}{n}^{-1}\binom{4n}{3n/2}\binom{5n}{n}\frac{1+(-1)^n}{2}.$$

We get for the LHS on multiplying by $\binom{4n}{n}$

$$\binom{3n}{k}\binom{5n+k}{4n}\binom{4n}{n} = \frac{(5n+k)!}{k! \times (3n-k)! \times (n+k)! \times n!}$$

$$= \binom{5n+k}{k}\binom{5n}{3n-k}\binom{2n+k}{n}.$$

Here we see that the middle coefficient enforces the upper range of the sum and we may extend k to infinity to get

$$[z^{3n}](1+z)^{5n}[w^n](1+w)^{2n} \sum_{k\geq 0} \binom{5n+k}{k} (-1)^k z^k (1+w)^k 2^{-k}$$

$$= [z^{3n}](1+z)^{5n}[w^n](1+w)^{2n} \frac{1}{(1+z(1+w)/2)^{5n+1}}$$

$$= 2^{5n+1}[z^{3n}](1+z)^{5n}[w^n](1+w)^{2n} \frac{1}{(2+z+zw)^{5n+1}}$$

$$= 2^{5n+1}[z^{8n+1}](1+z)^{5n}[w^n](1+w)^{2n}\frac{1}{(w+(2+z)/z)^{5n+1}}.$$

Here the contribution from w is

$$\operatorname*{res}_{w} \frac{1}{w^{n+1}}(1+w)^{2n}\frac{1}{(w+(2+z)/z)^{5n+1}}$$

and the residue at infinity is zero so we may evaluate using minus the residue at $w = -(2+z)/z$ which requires the Leibniz rule:

$$\frac{1}{(5n)!}\left(\frac{1}{w^{n+1}}(1+w)^{2n}\right)^{(5n)}$$

$$= \frac{1}{(5n)!}\sum_{q=0}^{5n}\binom{5n}{q}\frac{1}{w^{n+1+q}}(n+1)^{\overline{q}}(-1)^q(1+w)^{2n-(5n-q)}(2n)^{\underline{5n-q}}$$

$$= \sum_{q=0}^{5n}\frac{1}{w^{n+1+q}}\binom{n+q}{q}(-1)^q(1+w)^{q-3n}\binom{2n}{5n-q}.$$

Instantiating to z observing the sign:

$$(-1)^n\sum_{q=0}^{5n}\binom{n+q}{q}\frac{z^{n+1+q}}{(2+z)^{n+1+q}}(-1)^{q-3n}\frac{2^{q-3n}}{z^{q-3n}}\binom{2n}{5n-q}.$$

Recollecting what we have,

$$2^{2n+1}[z^{4n}](1+z)^{5n}\sum_{q=0}^{5n}\binom{n+q}{q}\frac{1}{(2+z)^{n+1+q}}(-1)^q 2^q\binom{2n}{5n-q}.$$

Here the contribution from z is

$$\operatorname*{res}_{z} \frac{1}{z^{4n+1}}(1+z)^{5n}\frac{1}{(z+2)^{n+1+q}}$$

and the residue at infinity is zero once again. With the Leibniz rule at $z = -2$:

$$\frac{1}{(n+q)!}\left(\frac{1}{z^{4n+1}}(1+z)^{5n}\right)^{(n+q)}$$

$$= \frac{1}{(n+q)!}\sum_{p=0}^{n+q}\binom{n+q}{p}\frac{1}{z^{4n+1+p}}(-1)^p(4n+1)^{\overline{p}}(1+z)^{5n-(n+q-p)}(5n)^{\underline{n+q-p}}.$$

Instantiating including the sign,

$$\sum_{p=0}^{n+q}\frac{1}{2^{4n+1+p}}\binom{4n+p}{p}(-1)^{q-p}\binom{5n}{n+q-p}.$$

Now we have

$$\binom{4n+p}{p}\binom{5n}{n+q-p} = \binom{5n}{n}\binom{4n+p}{4n+p-q}\binom{n+q}{p}\binom{n+q}{q}^{-1}.$$

Hence we find

$$2^{2n+1}\binom{5n}{n}\sum_{q=0}^{5n}\binom{2n}{5n-q}(-1)^q 2^q \sum_{p=0}^{n+q}\binom{4n+p}{q}\binom{n+q}{p}\frac{1}{2^{4n+1+p}}(-1)^{q-p}$$

$$= \frac{1}{2^{2n}}\binom{5n}{n}\sum_{q=0}^{5n}\binom{2n}{5n-q}2^q\sum_{p=0}^{n+q}\binom{4n+p}{q}\binom{n+q}{p}\frac{1}{2^p}(-1)^p.$$

We have for the inner sum including the 2^q

$$2^q\sum_{p=0}^{n+q}\binom{4n+p}{q}\binom{n+q}{p}\frac{1}{2^p}(-1)^p$$

$$= 2^q[z^q](1+z)^{4n}(1-(1+z)/2)^{n+q} = \frac{1}{2^n}[z^q](1+z)^{4n}(1-z)^{n+q}.$$

Returning to the main sum

$$\frac{1}{2^{3n}}\binom{5n}{n}\sum_{q=0}^{5n}\binom{2n}{5n-q}[z^q](1+z)^{4n}(1-z)^{n+q}$$

$$= \frac{1}{2^{3n}}\binom{5n}{n}\sum_{q=0}^{5n}\binom{2n}{q}[z^{5n-q}](1+z)^{4n}(1-z)^{6n-q}$$

$$= \frac{1}{2^{3n}}\binom{5n}{n}[z^{5n}](1+z)^{4n}(1-z)^{6n}\sum_{q=0}^{5n}\binom{2n}{q}\frac{z^q}{(1-z)^q}$$

$$= \frac{1}{2^{3n}}\binom{5n}{n}[z^{5n}](1+z)^{4n}(1-z)^{6n}\frac{1}{(1-z)^{2n}} = \frac{1}{2^{3n}}\binom{5n}{n}[z^{5n}](1-z^2)^{4n}.$$

We obtain zero when n is odd. We thus have

$$\frac{1+(-1)^n}{2}\frac{1}{2^{3n}}\binom{5n}{n}(-1)^{5n/2}\binom{4n}{5n/2}$$
$$= \frac{1+(-1)^n}{2}\frac{1}{2^{3n}}\binom{5n}{n}(-1)^{3n/2}\binom{4n}{3n/2}.$$

This is the claim. (Recall that we changed sides with the $\binom{4n}{n}$ factor.)

8 Identity 3.41 Vol 6

We seek the following identity:

$$\sum_{k=0}^{n} \binom{2x+1}{2k+1}\binom{x-k}{n-k} = 2^{2n}\frac{2x+1}{2n+1}\binom{x+n}{2n}.$$

Seeing that the left is a polynomial in x of degree $n+1+k$ i.e. $2n+1$ maximum as is the right it will suffice to prove the identity for x a negative integer $\le -2n$ to have it hold for complex x. Here the second binomial coefficient enforces the upper range and we find

$$[z^n](1+z)^x \sum_{k\ge 0} \binom{2x+1}{2k+1}\frac{z^k}{(1+z)^k}.$$

The sum is

$$\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{2x+1}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^x}{z^{n+1}}\sum_{k\ge 0}\frac{1}{w^{2k+2}}\frac{z^k}{(1+z)^k}\; dz\; dw.$$

For this to converge we need $|z/(1+z)/w^2| < 1$. Setting $\gamma = 1/2$ we require $|4z/(1+z)| < 1$. The bound is $4\varepsilon/(1-\varepsilon)$ so that $\varepsilon = 1/8$ will work. Continuing,

$$\frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{2x+1}\frac{1}{w^2}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^x}{z^{n+1}}\frac{1}{1-z/(1+z)/w^2}\; dw\; dz$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{2x+1}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{x+1}}{z^{n+1}}\frac{1}{w^2(1+z)-z}\; dz\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{2x+1}}{w^2-1}\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{(1+z)^{x+1}}{z^{n+1}}\frac{1}{z-w^2/(1-w^2)}\; dz\; dw.$$

With $x \le -2n$ we have no residue at infinity. We do have a pole at $z=-1$, which is outside the contour.

The pole at $w^2/(1-w^2) = -1 + 1/(1-w^2) = w^2 + w^4 + \cdots$ is contained in an annulus of radii $\gamma^2 \pm \frac{\gamma^4}{1-\gamma^2} = \frac{1}{4} \pm \frac{1}{12}$ so it is also outside the contour. It follows that we may evaluate using minus the contributions from the two finite poles and the residue at infinity. which is zero. We get from the simple pole,

$$-\frac{1}{2\pi i}\int_{|w|=\gamma}\frac{(1+w)^{2x+1}}{w^2-1}\frac{1}{(1-w^2)^{x+1}}\frac{(1-w^2)^{n+1}}{w^{2n+2}}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{2x+1}(1-w^2)^{n-x-1}\frac{1}{w^{2n+2}}\; dw$$

$$= \frac{1}{2\pi i}\int_{|w|=\gamma}(1+w)^{n+x}(1-w)^{n-x-1}\frac{1}{w^{2n+2}}\; dw.$$

The only pole other than zero here is at $w = -1$ and it is not inside the contour. The residue at infinity is zero. We get by upper negation

$$\sum_{q=0}^{2n+1} \binom{n+x}{2n+1-q} (-1)^q \binom{n-x-1}{q} = \sum_{q=0}^{2n+1} \binom{n+x}{2n+1-q} \binom{q+x-n}{q}.$$

Now we have

$$\begin{aligned}
\binom{n+x}{x-n+q-1}\binom{q+x-n}{q} &= \frac{(n+x)!}{(2n+1-q)! \times q! \times (x-n)!}(q+x-n) \\
&= \binom{x+n}{x-n}\binom{2n}{q}\frac{q+x-n}{2n+1-q} \\
&= \binom{x+n}{2n}\binom{2n}{q}\left[\frac{q-2n-1}{2n+1-q} + \frac{x-n+(2n+1)}{2n+1-q}\right] \\
&= \binom{x+n}{2n}\binom{2n}{q}\left[-1 + \frac{x+1+n}{2n+1-q}\right].
\end{aligned}$$

We get from the first piece,

$$-\binom{x+n}{2n} 2^{2n}$$

and the second one

$$(x+1-n)\binom{x+n}{2n}\frac{1}{2n+1}2^{2n+1}.$$

Adding the two pieces we find

$$\binom{x+n}{2n}2^{2n}\left[\frac{2(x+1+n)}{2n+1} - 1\right] = \binom{x+n}{2n}2^{2n}\frac{2x+1}{2n+1}.$$

We have achieved the desired closed form. It still remains to verify that the contribution from $z = -1$ is zero. This requires the Leibniz rule (keeping in mind that $x \le -2n$)

$$\begin{aligned}
&\frac{1}{(-x-2)!}\left(\frac{1}{z^{n+1}}\frac{1}{z - w^2/(1-w)^2}\right)^{(-x-2)} \\
&= \frac{1}{(-x-2)!}\sum_{q=0}^{-x-2}\binom{-x-2}{q}\frac{1}{z^{n+1+q}}(n+1)^{\overline{q}}(-1)^q \frac{1^{\overline{-x-2-q}}(-1)^{-x-2-q}}{(z-w^2/(1-w)^2)^{1+(-x-2-q)}} \\
&= (-1)^{-x}\sum_{q=0}^{-x-2}\frac{1}{z^{n+1+q}}\binom{n+q}{q}\frac{1}{(z-w^2/(1-w)^2)^{-x-1-q}}.
\end{aligned}$$

Instantiating $z=-1$ we have $-1-w^2/(1-w^2) = -1/(1-w^2)$ and obtain

$$(-1)^{-x+n+1} \sum_{q=0}^{-x-2} (-1)^q \binom{n+q}{q} (-1)^{-x-1-q} (1-w^2)^{-x-1-q}.$$

Now we observe that $-x-1-q \ge 1$ because $-x-2 \ge q$. Hence the factor (w^2-1) in the outer contour integral in w is canceled, removing the pole at $w=1$ which leaves no poles ar all and no new ones inside the contour, for a zero contribution and we may conclude.

9 Identity 5.16 Vol 6

We seek the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{2k}{k} \frac{1}{2^{2k}} \binom{m+k}{k}^{-1} = \frac{1}{2^{2n}} \binom{2m+2n}{m+n} \binom{2m}{m}^{-1}.$$

Re-index the left:

$$\frac{1}{2^{2n}} (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{2n-2k}{n-k} 2^{2k} \binom{m+n-k}{n-k}^{-1}.$$

Recall the inverse binomial coefficient identity

$$\binom{n-1}{k-1}^{-1} = n[v^n] \log \frac{1}{1-v} (v-1)^{n-k}.$$

Apply to obtain

$$\frac{1}{2^{2n}} (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{2n-2k}{n-k} 2^{2k}$$
$$\times (m+n-k+1) [v^{m+n-k+1}] \log \frac{1}{1-v} (v-1)^m$$
$$= \frac{1}{2^{2n}} (-1)^n [v^{m+n+1}] \log \frac{1}{1-v} (v-1)^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{2n-2k}{n-k} 2^{2k}$$
$$\times (m+n-k+1) v^k$$
$$= \frac{1}{2^{2n}} (-1)^n [v^{m+n+1}] \log \frac{1}{1-v} (v-1)^m [z^n] (1+z)^{2n}$$
$$\times \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{z^k}{(1+z)^{2k}} 2^{2k} (m+n-k+1) v^k.$$

This will produce two pieces.

First piece

$$(m+n+1)\frac{1}{2^{2n}}(-1)^n[v^{m+n+1}]\log\frac{1}{1-v}(v-1)^m[z^n](1+z)^{2n}\left[1-\frac{4zv}{(1+z)^2}\right]^n$$
$$=(m+n+1)\frac{1}{2^{2n}}(-1)^n[v^{m+n+1}]\log\frac{1}{1-v}(v-1)^m[z^n]\left[(1+z)^2-4zv\right]^n.$$

Expanding the powered term,

$$[(1+z)^2-4z(v-1)-4z]^n=[(1-z)^2-4z(v-1)]^n$$
$$=\sum_{q=0}^{n}\binom{n}{q}(1-z)^{2n-2q}(-1)^q4^qz^q(v-1)^q.$$

With the extractors

$$(m+n+1)\frac{1}{2^{2n}}[v^{m+n+1}]\log\frac{1}{1-v}(v-1)^m\sum_{q=0}^{n}\binom{n}{q}\binom{2n-2q}{n-q}4^q(v-1)^q$$
$$=\sum_{q=0}^{n}\binom{n}{q}\binom{2n-2q}{n-q}\frac{1}{4^{n-q}}\binom{m+n}{n-q}^{-1}.$$

Now we have

$$\binom{n}{q}\binom{m+n}{n-q}^{-1}=\frac{n!\times(m+q)!}{(m+n)!\times q!}=\binom{m+n}{n}^{-1}\binom{m+q}{m}.$$

We find for our sum,

$$\binom{m+n}{n}^{-1}\sum_{q=0}^{n}\binom{m+q}{m}\binom{2n-2q}{n-q}\frac{1}{4^{n-q}}$$
$$=\binom{m+n}{n}^{-1}[w^n]\frac{1}{(1-w)^{m+1}}\frac{1}{\sqrt{1-w}}=\binom{m+n}{n}^{-1}[w^n]\frac{1}{(1-w)^{m+3/2}}$$
$$=\binom{m+n}{n}^{-1}\binom{m+1/2+n}{n}.$$

Second piece

This piece is being subtracted:

$$-\frac{n}{2^{2n}}(-1)^n[v^{m+n+1}]\log\frac{1}{1-v}(v-1)^m[z^n](1+z)^{2n}$$
$$\times\sum_{k=1}^{n}\binom{n-1}{k-1}(-1)^k\frac{z^k}{(1+z)^{2k}}2^{2k}v^k$$

$$= \frac{4n}{2^{2n}}(-1)^n[v^{m+n}] \log\frac{1}{1-v}(v-1)^m[z^{n-1}](1+z)^{2n-2}$$

$$\times \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{z^k}{(1+z)^{2k}} 2^{2k} v^k$$

$$= \frac{4n}{2^{2n}(m+n)}(-1)^n(m+n)[v^{m+n}] \log\frac{1}{1-v}(v-1)^m$$

$$\times [z^{n-1}](1+z)^{2n-2}\left[1-\frac{4zv}{(1+z)^2}\right]^{n-1}.$$

Now if in the first piece we replace n by $n-1$ and multiply by $n/(m+n)$ we get the second piece. This yields

$$-\frac{n}{m+n}\binom{m+n-1}{n-1}^{-1}\binom{m+1/2+n-1}{n-1}$$

$$= -\binom{m+n}{n}^{-1}\binom{m+1/2+n-1}{n-1}.$$

Synthesis

Joining the two pieces we obtain

$$\binom{m+n}{n}^{-1}\left[\binom{m+1/2+n}{n} - \binom{m-1/2+n}{n-1}\right]$$

$$= \binom{m+n}{n}^{-1}\left[\binom{m+1/2+n}{n} - \frac{n}{n+m+1/2}\binom{m+1/2+n}{n}\right]$$

$$= \binom{m+n}{n}^{-1}\frac{m+1/2}{n+m+1/2}\binom{m+1/2+n}{n}$$

$$= \binom{m+n}{n}^{-1}\frac{2m+1}{2n+2m+1}\frac{1}{n!}(m+1/2+n)^{\underline{n}}.$$

We have for the falling factorial

$$\frac{1}{2^n}\frac{(2m+2n+1)!!}{(2m+1)!!} = \frac{1}{2^n}\frac{(2m+2n+1)!}{(2m+1)!}\frac{m!2^m}{(m+n)!2^{m+n}}.$$

Concluding with the simplification we obtain

$$\binom{m+n}{n}^{-1}\frac{1}{n!}\frac{1}{2^n}\frac{(2m+2n)!}{(2m)!}\frac{m!2^m}{(m+n)!2^{m+n}} = \frac{1}{2^{2n}}\binom{2m+2n}{m+n}\binom{2m}{m}^{-1}.$$

This is the claim.

The inverse binomial coefficient identity is from the paper "Inverse binomial coefficients in Egorychev method" [Mar25].

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