The cycle index of the set operator in the enumeration of identity trees

Marko R. Riedel

July 4, 2026

1 Counting rooted identity trees

In this document we will derive the recurrence for the number of rooted identity trees i.e. rooted trees whose automorphism group is the identity. These are precisely the unlabeled rooted trees where subtrees at a node are unique. Using the notation from *Analytic Combinatorics* by Flajolet and Sedgewick [FS09] we have for the corresponding combinatorial class the specification

$$\mathcal{T} = \mathcal{Z} \times \mathrm{PSET}(\mathcal{T})$$

Compare with the labeled specificaton which gives Cayley trees. We will derive a standard cycle index here for a self contained document. This material is from Theorem I.1 page 27 of [FS09].

2 Cycle index of the unlabeled set operator

Given an OGF $B(z)$ of a combinatorial class $\mathcal{B}$ where

$$B(z) = \sum_{\beta \in \mathcal{B}} z^{|\beta|}$$

we find for a set of n items drawn from $\mathcal{B}$ the OGF

$$[w^n] \prod_{\beta \in \mathcal{B}} (1 + w z^{|\beta|}) = [w^n] \exp\left(\sum_{\beta \in \mathcal{B}} \log(1 + w z^{|\beta|})\right)$$

$$= [w^n] \exp\left(\sum_{\beta \in \mathcal{B}} \sum_{\ell \ge 1} (-1)^{\ell-1} w^\ell z^{\ell|\beta|}/\ell\right) = [w^n] \exp\left(\sum_{\ell \ge 1} (-1)^{\ell-1} w^\ell B(z^\ell)/\ell\right).$$

Here we recognize the substitution from the Polya Enumeration Theorem and hence we obtain for the cycle index of the set operator $\mathrm{PSET}_{=n}$

$$\boxed{Z(P_n) = [w^n] \exp\left(\sum_{\ell \ge 1} (-1)^{\ell-1} a_\ell w^\ell/\ell\right).}$$

In the case where the set may be of any size we have $\mathrm{PSET}(\mathcal{B}) = \bigcup_{n \ge 0} \mathrm{PSET}_{=n}(\mathcal{B})$

so that we extract all coefficients on w and add the result, which is thus given by

$$P(z) = \exp\left(\sum_{\ell\ge 1} (-1)^{\ell-1} B(z^\ell)/\ell\right).$$

Observe that even if $\mathcal{B}$ includes items β of zero size counted by $b_0 = [z^0]B(z)$ these will formally produce a factor 2^{b_0} which corresponds to choosing whether or not they are included in the set, where they make no contribution to the total size of the set.

3 Recurrence relation for $Z(P_n)$

We have by differentiation when $n \ge 1$

$$Z(P_n) = \frac{1}{n}[w^{n-1}] \exp\left(\sum_{\ell\ge 1} (-1)^{\ell-1} a_\ell w^\ell/\ell\right) \sum_{\ell\ge 1} (-1)^{\ell-1} a_\ell w^{\ell-1}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} [w^k] \exp\left(\sum_{\ell\ge 1} (-1)^{\ell-1} a_\ell w^\ell/\ell\right) [w^{n-1-k}] \sum_{\ell\ge 1} (-1)^{\ell-1} a_\ell w^{\ell-1}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} (-1)^{n-1-k} a_{n-k} Z(P_k).$$

with base case $Z(P_0) = 1$. Here we obtain e.g. for sets of four elements the cycle index (memoize the recurrence)

$$Z(P_4) = \frac{1}{24}a_1^4 - \frac{1}{4}a_1^2 a_2 + \frac{1}{3}a_1 a_3 + \frac{1}{8}a_2^2 - \frac{1}{4}a_4.$$

It can be shown that $Z(P_n)$ is the cycle index $Z(S_n)$ of the symmetric group S_n under the substitution $a_\ell := (-1)^{\ell-1} a_\ell$. In this way we have constructed a cycle index for the set operator, generalizing the construction of additional admissible unlabeled classes from the cycle indices of the correspondig permutation group G permuting n elements through the OGF PET substitution $Z(G; B(z))$ (labeled classes use the cardinality $|G|$ through the EGF $F(z)^n/|G|$). For example, we may introduce the class $\text{DHD}_{=n}$ which implements dihedral symmetry and uses the cycle index $Z(D_n)$ of the dihedral group D_n, and appears e.g. when counting colored bracelets where turning over is allowed.

4 Recurrence relation for coefficients of $T(z)$

Using the cycle index we derived we obtain

$$T(z) = z \sum_{m\ge 0} Z(P_m; T(z)) = z \exp\left(\sum_{\ell\ge 1} (-1)^{\ell-1} T(z^\ell)/\ell\right).$$

Introducing $T_n = [z^n]T(z)$ we find from the functional equation

$$T_{n+1} = [z^n] \exp\left(\sum_{\ell\ge 1} (-1)^{\ell-1} T(z^\ell)/\ell\right).$$

Differentiation now yields

$$\frac{1}{n}[z^{n-1}] \exp\left(\sum_{\ell\ge 1}(-1)^{\ell-1} T(z^\ell)/\ell\right) \left(\sum_{\ell\ge 1}(-1)^{\ell-1} T'(z^\ell) z^{\ell-1}\right)$$
$$= \frac{1}{n}[z^{n-1}] \frac{T(z)}{z} \left(\sum_{\ell\ge 1}(-1)^{\ell-1} T'(z^\ell) z^{\ell-1}\right).$$

Actually doing the extraction with the Cauchy product

$$T_{n+1} = \frac{1}{n}\sum_{k=0}^{n-1} [z^{n-1-k}] \frac{T(z)}{z} [z^k] \sum_{\ell\ge 1}(-1)^{\ell-1} T'(z^\ell) z^{\ell-1}$$
$$= \frac{1}{n}\sum_{k=0}^{n-1} T_{n-k} [z^k] \sum_{\ell\ge 1}(-1)^{\ell-1} T'(z^\ell) z^{\ell-1}$$
$$= \frac{1}{n}\sum_{k=1}^{n} T_{n+1-k} [z^k] \sum_{\ell=1}^{n}(-1)^{\ell-1} T'(z^\ell) z^{\ell}.$$

Here we see that ℓ must divide k which yields

$$\frac{1}{n}\sum_{k=1}^{n} T_{n+1-k} [z^k] \sum_{\ell|k}(-1)^{\ell-1} T'(z^\ell) z^{\ell}$$
$$= \frac{1}{n}\sum_{k=1}^{n} T_{n+1-k} \sum_{\ell|k}(-1)^{\ell-1} [z^{\ell\times k/\ell}] T'(z^\ell) z^{\ell}$$
$$= \frac{1}{n}\sum_{k=1}^{n} T_{n+1-k} \sum_{\ell|k}(-1)^{\ell-1} [z^{k/\ell}] T'(z) z.$$

We have that

$$[z^{k/\ell}]T'(z)z = [z^{k/\ell-1}]T'(z) = \frac{k}{\ell} T_{k/\ell}$$

This finally gives for the recurrence with base case $T_1 = 1$

$$T_{n+1} = \frac{1}{n}\sum_{k=1}^{n} T_{n+1-k} \sum_{\ell|k}(-1)^{\ell-1} \frac{k}{\ell} T_{k/\ell}$$

or

$$\boxed{T_{n+1} = \frac{1}{n}\sum_{k=1}^{n} T_{n+1-k} \sum_{\ell|k}(-1)^{k/\ell-1} \ell T_{\ell}.}$$

Computing the recurrence yields the sequence

$$1, 1, 1, 2, 3, 6, 12, 25, 52, 113, 247, 548, 1226, 2770, 6299, 14426, \ldots$$

which is OEIS A004111.

5 Additional example: ternary and quaternary identity trees

If we have a bound q on the outdegree the combinatorial class becomes

$$\mathcal{T} = \mathcal{Z} \times \mathrm{PSET}_{\leq q}(\mathcal{T})$$

We may also consider the degree to be exact and pad with leaves of size zero of which there may be more than one. E.g. for $q = 4$ the cycle index contribution would be

$$Z(P_0) + Z(P_1) + Z(P_2) + Z(P_3) + Z(P_4)$$
$$= 1 + a_1 + \frac{1}{2}a_1^2 - \frac{1}{2}a_2 + \frac{1}{3}a_3 - \frac{1}{2}a_2a_1 + \frac{1}{6}a_1^3 - \frac{1}{4}a_4 + \frac{1}{3}a_3a_1 - \frac{1}{4}a_2a_1^2 + \frac{1}{8}a_2^2 + \frac{1}{24}a_1^4.$$

The functional equation using the PET substitution $a_\ell = T(z^\ell)$ is

$$T(z) = z(Z(P_0; T(z)) + Z(P_1; T(z)) + Z(P_2; T(z)) + Z(P_3; T(z)) + Z(P_4; T(z)))$$

or

$$T(z) = z\left(1 + T(z) + \frac{1}{2}T(z)^2 - \frac{1}{2}T(z^2) + \frac{1}{3}T(z^3) - \frac{1}{2}T(z)T(z^2) + \cdots + \frac{1}{24}T(z)^4\right).$$

We have for $q = 3$ the sequence

$$1, 1, 1, 2, 3, 6, 12, 25, 52, 113, 245, 542, 1205, 2707, 6113, 13907, \ldots$$

which is OEIS A116379 and for $q = 4$ the sequence

$$1, 1, 1, 2, 3, 6, 12, 25, 52, 113, 247, 548, 1226, 2770, 6298, 14419, \ldots$$

which is OEIS A116380.

References

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.