A Jordan totient sum

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February 2026

Problem statement

We seek to prove the following identity.

$$\zeta(r+1) = \frac{n^r}{J_r(n)} \sum_{m\ge 1} \frac{\mu(m)}{J_{r+1}(m)} \sum_{d|\gcd(m,n)} d \times \mu(m/d).$$

We refer to the usual number theoretic functions. Here $J_r(n)$ is the Jordan totient function. It is conjectured that this identity holds for complex r in the strip $\langle 1, \infty\rangle$ where the totient is defined for complex s by

$$J_s(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right).$$

First proof using Dirichlet series

We first evaluate

$$\mu(m) \sum_{d|\gcd(m,n)} d \times \mu(m/d).$$

We get zero from the first term when m is not a product of primes, so that's what we will henceforth assume. Then we write

$$\mu(m) \sum_{d|\gcd(m,n)} d \times \mu(\gcd(m,n)/d \times m/\gcd(m,n)).$$

Now in the second inner term the division by the gcd removes the primes that n shares with m. The first term has just those primes in the numerator. Hence we have a case where μ is multiplicative and we get

$$\mu(m)\mu(m/\gcd(m,n))\varphi(\gcd(m,n)).$$

We then find for the sum

$$\sum_{d|n} \sum_{\substack{m\ge 1\\(n,m)=1}} \frac{\mu(dm)\mu(m)\varphi(d)}{J_{r+1}(dm)} = \sum_{d|n} \varphi(d) \sum_{\substack{m\ge 1\\(n,m)=1}} \frac{\mu(dm)\mu(m)}{J_{r+1}(dm)}.$$

Here we take advantage of the property that $(d,m)=1$ to find

$$\sum_{d|n} \frac{\varphi(d)\mu(d)}{J_{r+1}(d)} \sum_{\substack{m\ge 1\\(n,m)=1}} \frac{\mu(m)^2}{J_{r+1}(m)}.$$

We introduce the Dirichlet series

$$L_{r+1}(s) = \sum_{\substack{m\ge 1\\(n,m)=1}} \frac{\mu(m)^2}{J_{r+1}(m)} \frac{1}{m^s} = \sum_{\substack{m\ge 1\\(n,m)=1}} \frac{\mu(m)^2}{J_{r+1}(m)/m^{r+1}} \frac{1}{m^{s+r+1}}.$$

Now if we did not have the condition on the gcd with n we would have the Euler product

$$\prod_p \left[1 + \frac{1}{1-1/p^{r+1}} \frac{1}{p^s}\right].$$

Evaluate at $s = r+1$ to get

$$\prod_p \left[1 + \frac{1}{1-1/p^{r+1}} \frac{1}{p^{r+1}}\right] = \prod_p \frac{1-1/p^{r+1}+1/p^{r+1}}{1-1/p^{r+1}} = \zeta(r+1).$$

We have to cancel those primes that divide n:

$$\zeta(r+1) \prod_{p|n} \left[1-1/p^{r+1}\right] = \zeta(r+1) \frac{J_{r+1}(n)}{n^{r+1}}.$$

Let us just recapitulate what we have at this point:

$$\zeta(r+1) \frac{J_{r+1}(n)}{n^{r+1}} \frac{n^r}{J_r(n)} \sum_{d|n} \frac{\varphi(d)\mu(d)}{J_{r+1}(d)}.$$

We thus need to show that

$$\prod_{p|n} \frac{1-1/p^{r+1}}{1-1/p^r} \sum_{d|n} \frac{\varphi(d)\mu(d)}{J_{r+1}(d)} = 1.$$

With the Dirichlet series

$$Q_{r+1}(s) = \sum_{n\ge 1} \frac{\varphi(n)\mu(n)}{J_{r+1}(n)} \frac{1}{n^s}$$

we have writing

$$\sum_{n\ge 1} \frac{\varphi(n)/n \times \mu(n)}{J_{r+1}(n)/n^{r+1}} \frac{1}{n^{s+r}}$$

the Euler product

$$\prod_p \left[1 - \frac{1-1/p}{1-1/p^{r+1}} \frac{1}{p^s}\right].$$

As we are summing these coefficients we multiply by $\zeta(s)$

$$\prod_p \frac{1}{1-p^{-s}} \prod_p \left[1 - \frac{1-1/p}{1-1/p^{r+1}} \frac{1}{p^{s+r}}\right]$$

This is

$$\prod_p \left[1 + \frac{p^{-s}}{1-p^{-s}} - \frac{1-1/p}{1-1/p^{r+1}} \frac{1}{p^r} \frac{p^{-s}}{1-p^{-s}}\right]$$

$$= \prod_p \left[1 + \left[1 - \frac{1-1/p}{1-1/p^{r+1}} \frac{1}{p^r}\right] \frac{p^{-s}}{1-p^{-s}}\right]$$

$$= \prod_p \left[1 + \frac{1-1/p^{r+1}-1/p^r+1/p^{r+1}}{1-1/p^{r+1}}\right] = \prod_p \left[1 + \frac{1-1/p^r}{1-1/p^{r+1}} \frac{p^{-s}}{1-p^{-s}}\right].$$

This is the claim and we may conclude the argument.

Second proof using a Lambert series

To start recall the series

$$\sum_{m\ge 1} \frac{\mu(m)}{m^r} \frac{x^m}{1-x^m}$$

which in $|x| < 1$ expands to

$$\sum_{m\ge 1} \frac{\mu(m)}{m^r} \sum_{q\ge 1} x^{mq} = \sum_{k\ge 1} x^k \sum_{m|k} \frac{\mu(m)}{m^r} = \sum_{k\ge 1} x^k \frac{1}{k^r} \sum_{m|k} \mu(m) \frac{k^r}{m^r} = \sum_{k\ge 1} \frac{J_r(k)}{k^r} x^k$$

so this is in fact the OGF of the scaled Jordan totient function $J_r(k)/k^r$. We now have from first principles that (let $S_{0,r} = 0$)

$$S_{n,r} = \sum_{k=1}^{n} \binom{n}{k} \frac{J_r(k)}{k^r} = \sum_{k=0}^{n-1} \binom{n}{n-k} [z^{n-k}] \sum_{m\ge 1} \frac{\mu(m)}{m^r} \frac{z^m}{1-z^m}.$$

Given that the series starts at z we may raise k to n:

$$\begin{aligned}[z^n] \sum_{m\ge 1} \frac{\mu(m)}{m^r} \frac{z^m}{1-z^m} \sum_{k=0}^{n} \binom{n}{k} z^k &= [z^n](1+z)^n \sum_{m\ge 1} \frac{\mu(m)}{m^r} \frac{z^m}{1-z^m} \\ &= \operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^n \sum_{m\ge 1} \frac{\mu(m)}{m^r} \frac{z^m}{1-z^m} \\ &= -\frac{1}{\zeta(r)} + \operatorname*{res}_z \frac{1}{z^{n+1}} (1+z)^n \sum_{m\ge 1} \frac{\mu(m)}{m^r} \frac{1}{1-z^m}.\end{aligned}$$

We only have convergence when $r \ge 2.$ Next we find using partial fractions by residues that

$$\begin{aligned}\frac{1}{1-z^m} &= \sum_{k=1}^{m} \frac{1}{z-\rho_{m,k}} \operatorname*{res}_{z=\rho_{m,k}} \frac{1}{1-z^m} \\ &= -\sum_{k=1}^{m} \frac{1}{z-\rho_{m,k}} \frac{1}{m\rho_{m,k}^{m-1}} = -\frac{1}{m} \sum_{k=1}^{m} \frac{\rho_{m,k}}{z-\rho_{m,k}}.\end{aligned}$$

where $\rho_{m,k} = \exp(2\pi i k/m)$ is a root of unity. We get for the Lambert series

$$\begin{aligned}&-\sum_{m\ge 1} \sum_{\substack{k=1\\(k,m)=1}}^{m} \frac{\rho_{m,k}}{z-\rho_{m,k}} \sum_{a\ge 1} \frac{\mu(am)}{(am)^{r+1}} \\ &= -\sum_{m\ge 1} \sum_{\substack{k=1\\(k,m)=1}}^{m} \frac{\rho_{m,k}}{z-\rho_{m,k}} \sum_{\substack{a\ge 1\\(m,a)=1}} \frac{\mu(a)\mu(m)}{(am)^{r+1}} \\ &= -\sum_{m\ge 1} \frac{\mu(m)}{m^{r+1}} \sum_{\substack{a\ge 1\\(m,a)=1}} \frac{\mu(a)}{a^{r+1}} \sum_{\substack{k=1\\(k,m)=1}}^{m} \frac{\rho_{m,k}}{z-\rho_{m,k}}.\end{aligned}$$

We have for the Dirichlet GF of the term in a that it is

$$\frac{1}{\zeta(s)} \prod_{p \text{ prime},\ p|m} \left(1 - \frac{1}{p^s}\right)^{-1}$$

so that we get

$$\frac{1}{\zeta(r+1)} \frac{m^{r+1}}{J_{r+1}(m)}$$

and for our sum

$$-\frac{1}{\zeta(r+1)} \sum_{m\ge 1} \frac{\mu(m)}{J_{r+1}(m)} \sum_{\substack{k=1\\(k,m)=1}}^{m} \frac{\rho_{m,k}}{z-\rho_{m,k}}.$$

Now we use the fact that residues sum to zero and the residue at infinity is zero by inspection, so we may evaluate using minus the residues at the roots of unity. When m is even the pole at $z=-1$ gets canceled. This is fine however as the factor $(1+z)^{n-1}$ becomes zero in that case. We obtain

$$\begin{aligned}&\frac{1}{\zeta(r+1)}\sum_{m\ge 1}\frac{\mu(m)}{J_{r+1}(m)}\sum_{\substack{k=1\\(k,m)=1}}^{m}\rho_{m,k}\frac{1}{\rho_{m,k}^{n+1}}(1+\rho_{m,k})^n\\ &=\frac{1}{\zeta(r+1)}\sum_{m\ge 1}\frac{\mu(m)}{J_{r+1}(m)}\sum_{\substack{k=1\\(k,m)=1}}^{m}\left[\frac{1+\rho_{m,k}}{\rho_{m,k}}\right]^n\\ &=\frac{1}{\zeta(r+1)}\sum_{m\ge 1}\frac{\mu(m)}{J_{r+1}(m)}\sum_{\substack{k=1\\(k,m)=1}}^{m}[1+\rho_{m,k}]^n.\end{aligned}$$

Working with the inner sum we have

$$\sum_{\substack{k=1\\(k,m)=1}}^{m}\sum_{q=0}^{n}\binom{n}{q}\rho_{m,k}^q=\sum_{q=0}^{n}\binom{n}{q}\sum_{\substack{k=1\\(k,m)=1}}^{m}\rho_{m,k}^q.$$

Next observe that

$$\sum_{d|m}\sum_{\substack{k=1\\(k,m)=d}}^{m}\rho_{m,k}^q=\sum_{k=1}^{m}\rho_{m,k}^q=m\times[[m|q]]$$

which implies

$$\sum_{\substack{k=1\\(k,m)=1}}^{m}\rho_{m,k}^q=\sum_{d|m}\mu(m/d)\times d\times[[d|q]]=\sum_{d|\ \gcd(m,q)}\mu(m/d)\times d.$$

We get for the sum

$$\frac{1}{\zeta(r+1)}\sum_{m\ge 1}\frac{\mu(m)}{J_{r+1}(m)}\sum_{q=1}^{n}\binom{n}{q}\sum_{d|\ \gcd(m,q)}\mu(m/d)\times d.$$

The contribution from $q=0$ is

$$\frac{1}{\zeta(r+1)}\sum_{m\ge 1}\frac{\mu(m)\varphi(m)}{J_{r+1}(m)}.$$

We have obtained an expansion in binomials where the constants do not depend on n. It is

$$S_{n,r}=-\frac{1}{\zeta(r)}+\frac{1}{\zeta(r+1)}\sum_{q=0}^{n}\binom{n}{q}\sum_{m\ge 1}\frac{\mu(m)}{J_{r+1}(m)}\sum_{d|\ \gcd(m,q)}d\times\mu(m/d).$$

Call the constants $\nu_{q,r}$ when $q\ge 1$ where we chose the index r because we are expanding $S_{n,r}$. Here we have supposed that $\gcd(m,0)=m$.

Inverting the sum

Recall from the first proof the value

$$Q_r(0)=\sum_{m\ge 1}\frac{\mu(m)\varphi(m)}{J_r(m)}$$

so that

$$\nu_{0,r}=Q_{r+1}(0)=\frac{\zeta(r+1)}{\zeta(r)}.$$

We may write for $k \ge 0$

$$\zeta(r+1)S_{k,r} + \frac{\zeta(r+1)}{\zeta(r)} + [[k=0]]\left(-\zeta(r+1)S_{k,r}\right) = \sum_{q=0}^{k} \binom{k}{q} \nu_{q,r}.$$

Summing in an inverse binomial transform we find on the right,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{q=0}^{k} \binom{k}{q} \nu_{q,r} = \sum_{q=0}^{n} \nu_{q,r} (-1)^n \sum_{k=q}^{n} \binom{n}{k} \binom{k}{q} (-1)^k.$$

With

$$\binom{n}{k}\binom{k}{q} = \frac{n!}{(n-k)! \times q! \times (k-q)!} = \binom{n}{q}\binom{n-q}{n-k}$$

we obtain

$$\begin{aligned} &\sum_{q=0}^{n} \binom{n}{q} \nu_{q,r} (-1)^n \sum_{k=q}^{n} \binom{n-q}{n-k} (-1)^k \\ &= \sum_{q=0}^{n} \binom{n}{q} \nu_{q,r} (-1)^{n+q} \sum_{k=0}^{n-q} \binom{n-q}{n-q-k} (-1)^k \\ &= \sum_{q=0}^{n} \binom{n}{q} \nu_{q,r} (-1)^{n+q} [[n-q=0]] = \nu_{n,r}. \end{aligned}$$

Applying the same on the left we get

$$\zeta(r+1)\frac{J_r(n)}{n^r} + \frac{\zeta(r+1)}{\zeta(r)}[[n=0]] + (-1)^n \times 0.$$

We thus have when $n \ge 1$

$$\nu_{n,r} = \zeta(r+1)\frac{J_r(n)}{n^r}.$$

Re-arrangeing we finally obtain

$$\zeta(r+1) = \frac{n^r}{J_r(n)} \sum_{m \ge 1} \frac{\mu(m)}{J_{r+1}(m)} \sum_{d \mid \gcd(m,n)} d \times \mu(m/d).$$

Interesting to note that the Zeta function value does not depend on n.