Partial fractions and Stirling numbers

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Plan of computation: we analyse a generating function in two ways: first as a rational generating function with poles at the inverses of the positive integers from the interval $[n]$ and second, as a generating function of the Stirling set numbers. We start with the partial fraction decomposition of

$$Q_{n,m}(z) = \prod_{r=1}^n \frac{1}{(1-rz)^m}.$$

Once the expansion into all the principal parts for each n is complete we can extract coefficients as follows:

$$[z^j] \frac{1}{(1-rz)^m} = r^j {j+m-1\choose j}.$$

We then sum over all principal parts. On the other hand we have

$$[z^j] Q_{n,m}(z) = [z^j] \left[\prod_{r=1}^n \frac{1}{1-rz}\right]^m$$

This is with $a_\ell \ge 0$

$$\sum_{a_1+a_2+\cdots+a_m=j} \prod_{\ell=1}^m {a_\ell+n\brace n}.$$

which means we obtain a closed form for the coefficient from the Stirling set numbers. To begin with we compute the Laurent series at $z=1/p$ and write

$$Q_{n,m}(z) = \frac{1}{p^m} (-1)^m \frac{1}{(z-1/p)^m} \prod_{r=1}^{p-1} \frac{1}{(1-rz)^m} \prod_{r=p+1}^n \frac{1}{(1-rz)^m}$$

we get for the first term

$$\begin{aligned} R_{n,m,0} &= \left.\prod_{r=1}^{p-1} \frac{1}{(1-rz)^m} \prod_{r=p+1}^n \frac{1}{(1-rz)^m}\right|_{z=1/p} \\ &= \prod_{r=1}^{p-1} \frac{1}{(1-r/p)^m} \prod_{r=p+1}^n \frac{1}{(1-r/p)^m} \\ &= p^{mn-m} \prod_{r=1}^{p-1} \frac{1}{(p-r)^m} \prod_{r=p+1}^n \frac{1}{(p-r)^m} \\ &= p^{m(n-1)} \frac{1}{(p-1)!^m} (-1)^{m(n-p)} \frac{1}{(n-p)!^m} \\ &= p^{m(n-1)} (-1)^{m(n-p)} \frac{1}{(n-1)!^m} {n-1\choose p-1}^m. \end{aligned}$$

We have for the second term, doing the differentiation,

$$\begin{aligned} R_{n,m,1} &= R_{n,m,0} \times \left.\left[\sum_{r=1}^{p-1} \frac{mr}{1-rz} + \sum_{r=p+1}^n \frac{mr}{1-rz}\right]\right|_{z=1/p} \\ &= R_{n,m,0} \times \left[\sum_{r=1}^{p-1} \frac{mpr}{p-r} + \sum_{r=p+1}^n \frac{mpr}{p-r}\right] \\ &= R_{n,m,0} \times mp \left[\sum_{r=1}^{p-1} \left(\frac{p}{p-r} - 1\right) + \sum_{r=p+1}^n \left(\frac{p}{r-p} - 1\right)\right] \\ &= R_{n,m,0} \times mp \left[-(n-1) + pH_{p-1} - pH_{n-p}\right]. \end{aligned}$$

For the remaining terms we need the derivatives of the bracketed term which we will call $P_{n,m,k}$. These derivatives are

$$\left[m \sum_{r=1}^{p-1} \frac{r^k}{(1-rz)^k}(k-1)! + m \sum_{r=p+1}^{n} \frac{r^k}{(1-rz)^k}(k-1)! \right]\Bigg|_{z=1/p}$$
$$= m(k-1)! \left[\sum_{r=1}^{p-1} \frac{r^k}{(1-r/p)^k} + \sum_{r=p+1}^{n} \frac{r^k}{(1-r/p)^k} \right]$$
$$= m(k-1)! p^k \left[\sum_{r=1}^{p-1} \frac{r^k}{(p-r)^k} + \sum_{r=p+1}^{n} \frac{r^k}{(p-r)^k} \right].$$

Evaluating the sums we find

$$\sum_{r=1}^{p-1} \left[\frac{p}{p-r} - 1 \right]^k = \sum_{r=1}^{p-1} \sum_{q=0}^{k} \binom{k}{q} \frac{p^q}{(p-r)^q} (-1)^{k-q}$$
$$= \sum_{q=0}^{k} \binom{k}{q} p^q (-1)^{k-q} \sum_{r=1}^{p-1} \frac{1}{(p-r)^q} = (-1)^k \sum_{q=0}^{k} \binom{k}{q} p^q (-1)^q H_{p-1}^{(q)}.$$

and repeating

$$\sum_{r=p+1}^{n} \left[\frac{p}{p-r} - 1 \right]^k = (-1)^k \sum_{q=0}^{k} \binom{k}{q} p^q H_{n-p}^{(q)}.$$

Joining these formulas we have

$$(-1)^k \sum_{q=0}^{k} \binom{k}{q} p^q ((-1)^q H_{p-1}^{(q)} + H_{n-p}^{(q)}).$$

Here we introduce the usual harmonic numbers given by

$$H_q(n) = H^{(q)}(n) = \sum_{r=1}^{n} \frac{1}{r^q}.$$

We are now ready to compute closed forms via the derivatives of $Q_{n,m}(z)$ which give the Laurent series and can be done symbolically. For example with $k = 4$ we find

$$Q(z)\, P_1(z)^4 + 6Q(z)\, P_1(z)^2 P_2(z) + 3Q(z)\, P_2(z)^2$$
$$+ 4Q(z)\, P_1(z)\, P_3(z) + Q(z)\, P_4(z)\,.$$

Here we have used

$$P_{n,m,k}(z) = \left(\sum_{r=1}^{p-1} \frac{mr}{1-rz} + \sum_{r=p+1}^{n} \frac{mr}{1-rz} \right)^{(k-1)}.$$

The key observation with this term is that its constituents do not increase in number as we differentiate and do not change form, the only change is in the coefficients and the exponent. An example of the partial fraction decompositions that we can now compute is

$$Q_{3,3}(z) = \frac{1}{8(1-z)^3} - \frac{64}{(1-2z)^3} + \frac{729}{8(1-3z)^3}$$
$$+ \frac{21}{16(1-z)^2} - \frac{384}{(1-2z)^2} - \frac{10935}{16(1-3z)^2}$$
$$+ \frac{129}{16\,(1-z)} - \frac{2112}{1-2z} + \frac{50301}{16\,(1-3z)}.$$

What we have with this decomposition is a closed form in terms of harmonic numbers and binomial coefficients, as opposed to numeric values, as will be seen below. For example with the coefficient on $1/(1-3z)^2$ we get symbolically

$$\frac{(-1)^m}{p^m} p^{m(n-1)} (-1)^{m(n-p)} \frac{1}{(n-1)!^m} \binom{n-1}{p-1}^m \times mp\left[-(n-1) + pH_{p-1} - pH_{n-p}\right]$$

which gives $-1215/16$ and indeed

$$-\frac{1215}{16}\frac{1}{(z-1/3)^2} = -\frac{10935}{16}\frac{1}{(1-3z)^2}.$$

The case of $m = 1$

Here we have one Stirling number in the product and we are extracting coefficients on $Q_{n,1}(z)$ which is the shifted Stirling set number OGF:

$$\sum_{a_1=j} \left\{ {a_1+n \atop n} \right\} = \frac{1}{(n-1)!} \sum_{p=1}^{n} (-1)^{n-p} p^{j+n-1} \binom{n-1}{p-1}$$

We can verify the correctness of this formula by evaluation:

$$\begin{aligned} &\frac{1}{(n-1)!}(j+n-1)![w^{j+n-1}] \sum_{p=0}^{n-1} (-1)^{n-1-p} \binom{n-1}{p} \exp((p+1)w) \\ &= \frac{1}{(n-1)!}(j+n-1)![w^{j+n-1}] \exp(w)(\exp(w)-1)^{n-1} \\ &= \frac{1}{n!}(j+n)![w^{j+n}](\exp(w)-1)^n = \left\{ {j+n \atop n} \right\}. \end{aligned}$$

This is as claimed.

The case of $m = 2$

$$\sum_{a_1+a_2=j} \left\{ {a_1+n \atop n} \right\} \left\{ {a_2+n \atop n} \right\}$$

$$= \frac{1}{(n-1)!^2} \sum_{p=1}^{n} p^{j+2n-2} \binom{n-1}{p-1}^2 \left[2pH(n-p) - 2pH(p-1) + j + 2n - 1 \right]$$

Note with all the formulas here we can compute the coefficient on $[z^j]$ as a sum of n terms, which relative to j is a constant.

The case of $m = 3$

$$\sum_{a_1+a_2+a_3=j} \left\{{a_1+n \atop n}\right\}\left\{{a_2+n \atop n}\right\}\left\{{a_3+n \atop n}\right\}$$
$$= \frac{1}{2(n-1)!^3} \sum_{p=1}^{n} (-1)^{n-p} p^{j+3n-3} \binom{n-1}{p-1}^3 \Big[9p^2 H(n-p)^2 + 9p^2 H(p-1)^2$$
$$+3p^2 H_2(n-p) + 3p^2 H_2(p-1)$$
$$+6\left(-3pH(p-1) + j + 3n - 1\right) pH(n-p) - 6\left(j+3n-1\right) pH(p-1)$$
$$+\left(j+3n-1\right)\left(j+3n-2\right) \Big]$$

The case of $m = 4$

$$\sum_{a_1+a_2+a_3+a_4=j} \left\{{a_1+n \atop n}\right\}\left\{{a_2+n \atop n}\right\}\left\{{a_3+n \atop n}\right\}\left\{{a_4+n \atop n}\right\}$$
$$= \frac{1}{6(n-1)!^4} \sum_{p=1}^{n} p^{j+4n-4} \binom{n-1}{p-1}^4 \Big[64p^3 H(n-p)^3 - 64p^3 H(p-1)^3$$
$$+8p^3 H_3(n-p) - 8p^3 H_3(p-1)$$
$$+48\left(-4pH(p-1) + j + 4n - 1\right) p^2 H(n-p)^2 + 48\left(j+4n-1\right) p^2 H(p-1)^2$$
$$+12\left(-4pH(p-1) + j + 4n - 1\right) p^2 H_2(n-p) + 12\left(j+4n-1\right) p^2 H_2(p-1)$$
$$+12\Big(16p^2 H(p-1)^2 + 4p^2 H_2(n-p) + 4p^2 H_2(p-1) - 8\left(j+4n-1\right) pH(p-1)$$
$$+\left(j+4n-1\right)\left(j+4n-2\right)\Big) pH(n-p)$$
$$-12\left(4p^2 H_2(p-1) + \left(j+4n-1\right)\left(j+4n-2\right)\right) pH(p-1)$$
$$+\left(j+4n-1\right)\left(j+4n-2\right)\left(j+4n-3\right) \Big]$$

The case of $m = 5$

$$\sum_{a_1+a_2+a_3+a_4+a_5=j} \left\{ {a_1+n \atop n} \right\} \left\{ {a_2+n \atop n} \right\} \left\{ {a_3+n \atop n} \right\} \left\{ {a_4+n \atop n} \right\} \left\{ {a_5+n \atop n} \right\}$$

$$= \frac{1}{24(n-1)!^5} \sum_{p=1}^{n} (-1)^{n-p} p^{j+5n-5} \binom{n-1}{p-1}^5 \Big[625 H(n-p)^4 p^4$$

$$+ 500p^3 \left(-5pH(p-1) + j + 5n - 1\right) H(n-p)^3$$

$$+150p^2 \Big(5p^2 H_2(n-p) + 25p^2 H(p-1)^2 - 10p\,(j+5n-1)\, H(p-1)$$

$$+ 5p^2 H_2(p-1) + (j+5n-1)\,(j+5n-2)\Big) H(n-p)^2$$

$$+ 20p \Big(15p^2 \left(-5pH(p-1) + j + 5n - 1\right) H_2(n-p) + 10p^3 H_3(n-p)$$

$$- 125p^3 H(p-1)^3 + 75p^2 \left(j+5n-1\right) H(p-1)^2$$

$$- 15p \left(5p^2 H_2(p-1) + (j+5n-1)\,(j+5n-2)\right) H(p-1) + 15p^2 \left(j+5n-1\right)$$

$$\times H_2(p-1) - 10p^3 H_3(p-1) + (j+5n-1)\,(j+5n-2)\,(5n-3+j)\Big) H(n-p)$$

$$+75 H_2(n-p)^2 p^4$$

$$+ 30p^2 \Big(25p^2 H(p-1)^2 - 10p\,(j+5n-1)\, H(p-1) + 5p^2 H_2(p-1)$$

$$+ (j+5n-1)\,(j+5n-2)\Big) H_2(n-p)$$

$$+ 40p^3 \left(-5pH(p-1) + j + 5n - 1\right) H_3(n-p) + 30p^4 H_4(n-p)$$

$$+625 H(p-1)^4 p^4 - 500p^3 \left(j+5n-1\right) H(p-1)^3$$

$$+ 150p^2 \left(5p^2 H_2(p-1) + (j+5n-1)\,(j+5n-2)\right) H(p-1)^2$$

$$- 20p \left(15p^2 \left(j+5n-1\right) H_2(p-1)\right.$$

$$\left. - 10p^3 H_3(p-1) + (j+5n-1)\,(j+5n-2)\,(5n-3+j)\right) H(p-1)$$

$$+ 75 H_2(p-1)^2 p^4 + 30p^2 \left(j+5n-1\right)\left(j+5n-2\right) H_2(p-1)$$

$$- 40p^3 \left(j+5n-1\right) H_3(p-1) + 30p^4 H_4(p-1)$$

$$+ (j+5n-1)\,(j+5n-2)\,(j+5n-3)\,(j+5n-4) \Big]$$