Six identities by M. Spivey, Egorychev method

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We apply Egorychev method to six identities from the book "The art of proving binomial coefficient identities" [Spi19] by M. Spivey.

Contents

List of identities in this document

section 1

$$\sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = 16^n \frac{1}{2n+1} \binom{2n}{n}^{-1}$$

section 2

$$\sum_{k=0}^{n} \binom{m}{k} \binom{n}{r-k} k^p = \binom{m+n}{r} \sum_{j=0}^{p} \left\{ {p \atop j} \right\} \frac{r^{\underline{j}} m^{\underline{j}}}{(m+n)^{\underline{j}}}$$

section 3

$$\sum_{k \ge r} \binom{k-1}{r-1} p^r (1-p)^{k-r} k^m = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} (-1)^{m-j} \frac{r^{\overline{j}}}{p^j}$$

section 4

$$\sum_{k=0}^{n}\sum_{j=0}^{m}\left[{n \atop k}\right]\binom{m}{j}j^k(-1)^{n-k}=m^{\underline{n}}2^{m-n}$$

section 5

$$\sum_{k=0}^{n}\binom{2k+r}{k}\binom{2n-2k}{n-k}\frac{r}{2k+r}=\binom{2n+r}{n}$$

section 6

$$\sum_{n\geq 0}(-1)^n\binom{2n}{n}^{-1}=\frac{4}{5}-\frac{4\sqrt{5}}{25}\log\frac{1+\sqrt{5}}{2}$$

1 Identity 166

We seek the following identity

$$\sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = 16^n \frac{1}{2n+1} \binom{2n}{n}^{-1}.$$

Re-indexing we find

$$\sum_{k=0}^{n} \frac{1}{2n-2k+1} \binom{2k}{k} \binom{2n-2k}{n-k}$$
$$= [v^{2n+1}] \log \frac{1}{1-v} \sum_{k=0}^{n} v^{2k} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Here the second binomial coefficient enforces the upper range and we may write for the remaining sum

$$[z^n](1+z)^{2n} \sum_{k\ge 0} v^{2k} \binom{2k}{k} \frac{z^k}{(1+z)^{2k}} = [z^n](1+z)^{2n} \frac{1}{\sqrt{1-4v^2z/(1+z)^2}}$$
$$= [z^n](1+z)^{2n+1} \frac{1}{\sqrt{(1+z)^2-4v^2z}} = [z^n](1+z)^{2n+1} \frac{1}{\sqrt{(1-z)^2-4(v^2-1)z}}$$
$$= [z^n] \frac{(1+z)^{2n+1}}{1-z} \frac{1}{\sqrt{1-4(v^2-1)z/(1-z)^2}}.$$

With the extractor in z we may expand the square root into a finite sum:

$$[z^n] \frac{(1+z)^{2n+1}}{1-z} \sum_{q=0}^{n} \binom{2q}{q} \frac{(v^2-1)^q z^q}{(1-z)^{2q}}$$
$$= [z^n](1+z)^{2n+1} \sum_{q=0}^{n} \binom{2q}{q} \frac{z^q}{(1-z)^{2q+1}} \sum_{p=0}^{q} \binom{q}{p} (-1)^{q-p} v^{2p}.$$

With the extractor in v,

$$[z^n](1+z)^{2n+1} \sum_{q=0}^{n} \binom{2q}{q} \frac{z^q}{(1-z)^{2q+1}} \sum_{p=0}^{q} \binom{q}{p} (-1)^{q-p} \frac{1}{2n+1-2p}.$$

Now for the inner sum introduce

$$F(z) = q! \frac{1}{2n+1-2z} \prod_{r=0}^{q} \frac{1}{z-r}.$$

This has the property that with $0 \le p \le q$

$$\operatorname*{res}_{z=p} F(z) = q! \frac{1}{2n+1-2p} \prod_{r=0}^{p-1} \frac{1}{p-r} \prod_{r=p+1}^{q} \frac{1}{p-r}$$

$$= q! \frac{1}{2n+1-2p} \frac{1}{p!} (-1)^{q-p} \frac{1}{(q-p)!} = \frac{1}{2n+1-2p} \binom{q}{p} (-1)^{q-p}.$$

Residues sum to zero and the residue at infinity is zero, hence we may sum by minus the residue at $z = n + 1/2$:

$$\frac{1}{2} q! \prod_{r=0}^{q} \frac{1}{n+1/2-r} = 2^q q! \prod_{r=0}^{q} \frac{1}{2n+1-2r}$$

$$= 2^q q! \frac{1}{2n-2q+1} \frac{(2n-2q+1)!!}{(2n+1)!!}$$

$$= 2^q q! \frac{2^n n!}{(2n+1)!} \frac{1}{2n-2q+1} \frac{(2n-2q+1)!}{2^{n-q}(n-q)!}$$

$$= 2^{2q} \binom{2n-2q}{n-q} \frac{1}{2n+1} \binom{2n}{n}^{-1} \binom{n}{q}^{-1}.$$

Re-writing the binomial coefficients,

$$\binom{2q}{q} \binom{n}{q}^{-1} \binom{2n-2q}{n-q} \binom{2n}{n}^{-1} = \binom{n}{q}^{-1} \binom{n}{q}^{2} \binom{2n}{2q}^{-1} = \binom{n}{q} \binom{2n}{2q}^{-1}.$$

We are now left with the following terms:

$$[z^n](1+z)^{2n+1} \sum_{q=0}^{n} \binom{n}{q} 2^{2q} \frac{1}{2n+1} \binom{2n}{2q}^{-1} \frac{z^q}{(1-z)^{2q+1}}.$$

Computation of auxiliary coefficient

For the term in z we find

$$[z^{n-q}](1+z)^{2n+1} \frac{1}{(1-z)^{2q+1}} = \operatorname*{res}_{z} \frac{1}{z^{n-q+1}} (1+z)^{2n+1} \frac{1}{(1-z)^{2q+1}}.$$

Now put $z/(1+z) = u$ so that $z = u/(1-u)$ to get

$$\operatorname*{res}_{u} \frac{1}{u^{n-q+1}} \frac{1}{(1-u)^{n+q}} \frac{(1-u)^{2q+1}}{(1-2u)^{2q+1}} \frac{1}{(1-u)^2}$$

$$= \operatorname*{res}_{u} \frac{1}{u^{n-q+1}} \frac{1}{(1-u)^{n-q+1}} \frac{1}{(1-2u)^{2q+1}}.$$

Next we put $u = (1-\sqrt{1-4w})/2$ so that $u(1-u) = w$ and $du = \frac{1}{\sqrt{1-4w}}\; dw$ to obtain

$$\operatorname*{res}_{w} \frac{1}{w^{n-q+1}} \frac{1}{\sqrt{1-4w}^{2q+1}} \frac{1}{\sqrt{1-4w}}$$
$$= \operatorname*{res}_{w} \frac{1}{w^{n-q+1}} \frac{1}{(1-4w)^{q+1}} = \binom{n-q+q}{q} 4^{n-q} = \binom{n}{q} 2^{2n-2q}.$$

Return to main sum

Collecting everything we gather

$$\frac{2^{2n}}{2n+1} \sum_{q=0}^{n} \binom{n}{q}^2 \binom{2n}{2q}^{-1} = \frac{2^{2n}}{2n+1} \binom{2n}{n}^{-1} \sum_{q=0}^{n} \binom{2q}{q} \binom{2n-2q}{n-q}$$
$$= \frac{2^{2n}}{2n+1} \binom{2n}{n}^{-1} [w^n] \frac{1}{\sqrt{1-4w}} \frac{1}{\sqrt{1-4w}} = \frac{2^{2n}}{2n+1} \binom{2n}{n}^{-1} [w^n] \frac{1}{1-4w}$$
$$= \frac{2^{2n}}{2n+1} \binom{2n}{n}^{-1} 4^n = \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-1}.$$

This is the claim and we may conclude.

2 Identity 252

We seek the following identity with $p, r \ge 0$

$$\sum_{k=0}^{n} \binom{m}{k} \binom{n}{r-k} k^p = \binom{m+n}{r} \sum_{j=0}^{p} \left\{ {p \atop j} \right\} \frac{r^{\underline{j}} m^{\underline{j}}}{(m+n)^{\underline{j}}}.$$

We have clearing the upper range

$$[z^n] \frac{1}{1-z} \sum_{k\ge 0} \binom{m}{k} \binom{n}{r-k} k^p z^p$$
$$= p! [z^n] \frac{1}{1-z} [w^p] \sum_{k\ge 0} \binom{m}{k} \binom{n}{r-k} \exp(kw) z^p$$
$$= p! [z^n] \frac{1}{1-z} [w^p][v^r] (1+v)^n \sum_{k\ge 0} \binom{m}{k} \exp(kw) z^p v^k$$
$$= p! [z^n] \frac{1}{1-z} [w^p][v^r] (1+v)^n (1+\exp(w) z v)^m.$$

Re-writing the powered term

$$p![z^n]\frac{1}{1-z}[w^p][v^r](1+v)^n \sum_{j=0}^m \binom{m}{j} (\exp(w)-1)^j z^j v^j (1+zv)^{m-j}.$$

We have for the inner sum,

$$\sum_{j=0}^m \binom{m}{j} (\exp(w)-1)^j z^j v^j \sum_{q=0}^{m-j} \binom{m-j}{q} z^q v^q.$$

With the extractor in v,

$$\sum_{j=0}^m \binom{m}{j} (\exp(w)-1)^j \sum_{q=0}^{m-j} \binom{m-j}{q} \binom{n}{r-q-j} z^{q+j}.$$

By construction the third binomial coefficient is zero when $r < q+j$. Continuing with the extractor in w

$$\sum_{j=0}^m m^{\underline{j}} \left\{ {p \atop j} \right\} \sum_{q=0}^{m-j} \binom{m-j}{q} \binom{n}{r-q-j} z^{q+j}.$$

The extractor in z now implements $q+j \le n$ or $q \le n-j$. Therefore if $n \ge m$ the entire sum is preserved and we get

$$\sum_{j=0}^m m^{\underline{j}} \left\{ {p \atop j} \right\} \sum_{q=0}^{m-j} \binom{m-j}{q} \binom{n}{r-q-j}.$$

This is

$$\sum_{j=0}^m m^{\underline{j}} \left\{ {p \atop j} \right\} \operatorname*{res}_z \frac{1}{z^{r-j+1}} (1+z)^n \sum_{q=0}^{m-j} \binom{m-j}{q} z^q$$

$$= \sum_{j=0}^m m^{\underline{j}} \left\{ {p \atop j} \right\} \operatorname*{res}_z \frac{1}{z^{r-j+1}} (1+z)^{n+m-j}$$

$$= \sum_{j=0}^m m^{\underline{j}} \left\{ {p \atop j} \right\} \binom{m+n-j}{r-j}$$

$$= \sum_{j=0}^m m^{\underline{j}} \left\{ {p \atop j} \right\} \binom{m+n-j}{m+n-r}.$$

In the remaining binomial coefficient we may suppose that $m+n \ge r$ or else it will produce zero for all j for a total of zero just like in the closed form. Here we may set the upper range to p. Supposing that $p > m$ the extra values are zero by the falling factorial with base m. On the other hand if $p < m$ the canceled values are zero from the Stirling number. Simplifying the closed form we find

$$\binom{m+n}{r}\sum_{j=0}^{p}\left\{{p \atop j}\right\}\binom{r}{j}j!\binom{m}{j}j!\binom{m+n}{j}^{-1}\frac{1}{j!}$$

$$=\sum_{j=0}^{p}\left\{{p \atop j}\right\}\binom{r}{j}\binom{m}{j}j!\frac{j!(m+n-j)!}{r!(m+n-r)!}$$

$$=\sum_{j=0}^{p}\left\{{p \atop j}\right\}m^{\underline{j}}\binom{m+n-j}{m+n-r}$$

and we have the claim.

3 Identity 253

We seek the following identity

$$\sum_{k\ge r}\binom{k-1}{r-1}p^r(1-p)^{k-r}k^m=\sum_{j=0}^{m}\left\{{m \atop j}\right\}(-1)^{m-j}\frac{r^{\overline{j}}}{p^j}.$$

Here we have the probability p being $0<p<1$ and r and m positive integers. We start by re-writing

$$p^r\sum_{k\ge 0}\binom{k+r-1}{r-1}(1-p)^k(k+r)^m$$

$$=p^r m![w^m]\exp(rw)\sum_{k\ge 0}\binom{k+r-1}{r-1}(1-p)^k\exp(kw)$$

$$=p^r m![w^m]\frac{\exp(rw)}{(1-(1-p)\exp(w))^r}$$

$$=p^r m![w^m]\frac{1}{(\exp(-w)-(1-p))^r}$$

$$=(-1)^m p^r m![w^m]\frac{1}{(\exp(w)-1+p)^r}$$

$$=(-1)^m m![w^m]\frac{1}{(1+(\exp(w)-1)/p)^r}.$$

Seeing that $\exp(w)-1=w+\cdots$ we may expand the geometric series again,

$$(-1)^m m![w^m]\sum_{j=0}^{m}\binom{j+r-1}{r-1}(-1)^j\frac{(\exp(w)-1)^j}{p^j}$$

$$= (-1)^m \sum_{j=0}^{m} \binom{j+r-1}{r-1} j!(-1)^j \frac{1}{p^j} \left\{ {m \atop j} \right\}.$$

Now we have

$$\binom{j+r-1}{r-1} \times j! = \binom{j+r-1}{j} \times j! = r^{\overline{j}}$$

which is the claim.

4 Identity 261

We seek the following identity

$$\sum_{k=0}^{n} \sum_{j=0}^{m} \left[{n \atop k} \right] \binom{m}{j} j^k (-1)^{n-k} = m^{\underline{n}} 2^{m-n}.$$

We find for the inner sum,

$$\sum_{j=0}^{m} \binom{m}{j} j^k = k![w^k] \sum_{j=0}^{m} \binom{m}{j} \exp(jw) = k![w^k](\exp(w)+1)^m.$$

Continuing with the outer sum

$$\sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k} \right] k![w^k](\exp(w)+1)^m$$

$$= n![z^n] \sum_{k=0}^{n} (-1)^{n-k} \frac{1}{k!} \left(\log \frac{1}{1-z} \right)^k k![w^k](\exp(w)+1)^m$$

$$= n![z^n] \sum_{k=0}^{n} (-1)^{n-k} \left(\log \frac{1}{1-z} \right)^k [w^k](\exp(w)+1)^m.$$

Now with $\log \frac{1}{1-z} = z + \cdots$ we get a zero contribution from the extractor in z when $k > n$ and we may raise the upper range to infinity:

$$(-1)^n n![z^n] \sum_{k \ge 0} (-1)^k \left(\log \frac{1}{1-z} \right)^k [w^k](\exp(w)+1)^m$$

$$= (-1)^n n![z^n] \left(\exp\left(-\log \frac{1}{1-z} \right) + 1 \right)^m = (-1)^n n![z^n](2-z)^m$$

$$= (-1)^n n! (-1)^n \binom{m}{n} 2^{m-n} = m^{\underline{n}} 2^{m-n}.$$

We have the claim.

5 Identity 278

We seek the following identity with $r \geq 1$

$$\sum_{k=0}^{n} \binom{2k+r}{k} \binom{2n-2k}{n-k} \frac{r}{2k+r} = \binom{2n+r}{n}.$$

Re-write,

$$\sum_{k=0}^{n} \binom{2k+r}{k+r} \binom{2n-2k}{n-k} \frac{r}{2k+r}$$
$$= \sum_{k=0}^{n} \frac{r}{k+r} \binom{2k+r-1}{k+r-1} \binom{2n-2k}{n-k}$$
$$= \sum_{k=0}^{n} \left[1 - \frac{k}{k+r}\right] \binom{2k+r-1}{k+r-1} \binom{2n-2k}{n-k}.$$

The first piece here is

$$\sum_{k=0}^{n} \binom{2k+r-1}{k+r-1} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} \binom{2n-2k+r-1}{n-k} \binom{2k}{k}.$$

The first binomial coefficient enforces the upper range of the sum and we get

$$[z^n](1+z)^{2n+r-1} \sum_{k\geq 0} \binom{2k}{k} \frac{z^k}{(1+z)^{2k}} = [z^n](1+z)^{2n+r-1} \frac{1}{\sqrt{1-4z/(1+z)^2}}$$
$$= [z^n](1+z)^{2n+r} \frac{1}{\sqrt{(1+z)^2-4z}} = [z^n](1+z)^{2n+r} \frac{1}{1-z}.$$

The second piece is

$$\sum_{k=1}^{n} \binom{2k+r-1}{k+r} \binom{2n-2k}{n-k} = \sum_{k=0}^{n-1} \binom{2n-2k+r-1}{n-1-k} \binom{2k}{k}.$$

Repeat to find

$$[z^{n-1}](1+z)^{2n+r-1} \sum_{k\geq 0} \binom{2k}{k} \frac{z^k}{(1+z)^{2k}} = [z^{n-1}](1+z)^{2n+r} \frac{1}{1-z}$$
$$= [z^n](1+z)^{2n+r} \frac{z}{1-z}.$$

Subtract the second piece from the first to get

$$[z^n](1+z)^{2n+r}\left[\frac{1}{1-z} - \frac{z}{1-z}\right] = [z^n](1+z)^{2n+r} = \binom{2n+r}{n}.$$

We have the claim.

6 Identity 328

First identity 328

We seek the following identity:

$$\sum_{n\ge 0} (-1)^n \binom{2n}{n}^{-1} = \frac{4}{5} - \frac{4\sqrt{5}}{25}\log\frac{1+\sqrt{5}}{2}.$$

Re-call the inverse binomial coefficient identity

$$\frac{1}{k}\binom{n}{k}^{-1} = [v^n]\log\frac{1}{1-v}(v-1)^{n-k}.$$

We can re-write this as

$$\binom{n-1}{k-1}^{-1} = n[v^n]\log\frac{1}{1-v}(v-1)^{n-k}$$
$$= n \operatorname*{res}_v \frac{1}{v^{n+1}}\log\frac{1}{1-v}(v-1)^{n-k}.$$

Now in formal power series we put $\log\frac{1}{1-v} = u$ so that $v = 1-\exp(-u)$ to get (here zero maps to zero)

$$n \operatorname*{res}_u \frac{1}{(1-\exp(-u))^{n+1}} u(-1)^{n-k}\exp(-u(n-k))\exp(-u)$$
$$= (-1)^{n-k} n \operatorname*{res}_u \frac{u\exp(-u(n+1-k))}{(1-\exp(-u))^{n+1}}.$$

The nearest poles are at $2\pi ik$ so this becomes with $\varepsilon < 2\pi$

$$\frac{(-1)^{n-k}n}{2\pi i}\int_{|u|=\varepsilon}\frac{u\exp(-u(n+1-k))}{(1-\exp(-u))^{n+1}}\;du.$$

We may certainly instantiate ε to $R = 3/2$. This gives for the sum

$$\sum_{n\ge 0}(-1)^n(-1)^{2n+1-(n+1)}$$
$$\times(2n+1)\frac{1}{2\pi i}\int_{|u|=R}\frac{u}{(1-\exp(-u))^{2n+2}}\exp(-u(2n+2-(n+1)))\;du$$

$$= \sum_{n\ge 0} (2n+1) \frac{1}{2\pi i} \int_{|u|=R} \frac{u}{(1-\exp(-u))^{2n+2}} \exp(-u(n+1)) \; du$$

$$= \sum_{n\ge 0} (2n+1) \frac{1}{2\pi i} \int_{|u|=R} u \left[\frac{\exp(u)}{(\exp(u)-1)^2}\right]^{n+1} \; du.$$

For the geometric series to converge we need $|\exp(u)/(\exp(u)-1)^2| < 1$. Now we have

$$[u^n] \frac{\exp(u)}{(\exp(u)-1)^2} = -(n+1)[u^{n+1}] \frac{1}{\exp(u)-1}$$
$$= -(n+1)[u^{n+2}] \frac{u}{\exp(u)-1}.$$

This is the EGF of the familiar Bernoulli numbers. The norm of the coefficients of this EGF is bounded by $1/(2\pi)^{n-1}$ for all n through the dominant singularity at $u = \pm 2\pi i$. The original pole at zero has

$$\frac{1}{u^2} \frac{\exp(u)}{((\exp(u)-1)/u)^2} = \frac{1}{u^2} \frac{\exp(u)}{(1+u/2+\cdots)^2}$$

so on the circle $|u| = R$ we get the bound for the norm

$$\frac{1}{R^2} + \frac{1}{12} + \sum_{n\ge 1} (n+1) \frac{R^{n+2}}{(2\pi)^{n+1}}$$

$$= \frac{1}{R^2} + \frac{1}{12} + R \sum_{n\ge 2} n \frac{R^n}{(2\pi)^n} = \frac{1}{R^2} + \frac{1}{12} + R\left[-\frac{R}{2\pi} + \frac{R/2/\pi}{(1-R/2/\pi)^2}\right].$$

At $R = 3/2$ this works out to $0.7875930684 < 1$ and we may go ahead and sum the geometric series. Putting $w = \exp(u) - 1$ we get two pieces, the first is (here we sum in u, before we make the substitution)

$$u \sum_{n\ge 0} \left[\frac{w+1}{w^2}\right]^{n+1} = u \frac{(w+1)/w^2}{1-(w+1)/w^2} = \frac{u(w+1)}{w^2-w-1}$$

and

$$2u \sum_{n\ge 0} n \left[\frac{w+1}{w^2}\right]^{n+1} = 2u \frac{(w+1)^2/w^4}{(1-(w+1)/w^2)^2} = \frac{2u(w+1)^2}{(w^2-w-1)^2}.$$

The poles here are at $w = (1 \pm \sqrt{5})/2 = \rho_\pm$. There is also a pole from $u = \log(1+w)$ at $w = -1$. Note that the image of the circle $|u| = R$ is by the series $\exp(u) - 1 = u + u^2/2 + u^3/6 + \cdots$ contained in a circle with radius $R + (\exp(R) - 1 - R)$. It makes one turn which comes from the u term. Note also that the real part of the curve $\exp(R \exp i\theta) - 1$ is given by (here $0 \le \theta < 2\pi$)

$$\exp(R\cos\theta) \times \cos(R\sin\theta) - 1.$$

This means the pole at $w = -1$ is not inside the contour because with $R = 3/2 < \pi/2$ the second cosine term is always positive, as is the first factor. On the other hand the remaining two poles are inside the image contour which attains its maximum distance from the origin on the real axis at θ at $\exp(R)-1 \approx 3.481689070 > \frac{1+\sqrt{5}}{2}$. The minimum distance on the real axis happens when the norm of $\exp(R\cos\theta)$ in minimal which is $\cos\theta = -1$ or $\theta = \pi$ and gives $\exp(-R) - 1 \approx -0.7768698399 < \frac{1-\sqrt{5}}{2}$. Continuing we get from the two poles (note that $du = \frac{1}{1+w}\,dw$), first,

$$\log(1+\rho_\pm)\frac{1}{2\rho_\pm - 1}.$$

which gives

$$\frac{1}{\sqrt{5}}\log\frac{3+\sqrt{5}}{2} - \frac{1}{\sqrt{5}}\log\frac{3-\sqrt{5}}{2}$$

and second

$$\left(2\log(1+w)\frac{1+w}{(w-\rho_\mp)^2}\right)'\Bigg|_{w=\rho_\pm}$$

$$= \left(2\frac{1}{(w-\rho_\mp)^2} + 2\log(1+w)\frac{1}{(w-\rho_\mp)^2} - 4\log(1+w)(1+w)\frac{1}{(w-\rho_\mp)^3}\right)'\Bigg|_{w=\rho_\pm}.$$

This gives

$$\frac{4}{5} + \frac{2}{5}\log\frac{3+\sqrt{5}}{2} + \frac{2}{5}\log\frac{3-\sqrt{5}}{2}$$
$$-4\frac{1}{5\sqrt{5}}\log\frac{3+\sqrt{5}}{2}\frac{3+\sqrt{5}}{2} + 4\frac{1}{5\sqrt{5}}\log\frac{3-\sqrt{5}}{2}\frac{3-\sqrt{5}}{2}.$$

Collecting everything we find

$$\frac{4}{5} + \log\frac{3+\sqrt{5}}{2}\left[\frac{2\sqrt{5}}{5\sqrt{5}} - \frac{4}{5\sqrt{5}}\frac{3+\sqrt{5}}{2} + \frac{5}{5\sqrt{5}}\right]$$
$$+\log\frac{3-\sqrt{5}}{2}\left[\frac{2\sqrt{5}}{5\sqrt{5}} + \frac{4}{5\sqrt{5}}\frac{3-\sqrt{5}}{2} - \frac{5}{5\sqrt{5}}\right]$$
$$= \frac{4}{5} + \log\frac{3+\sqrt{5}}{2}\left[-\frac{1}{5\sqrt{5}}\right] + \log\frac{3-\sqrt{5}}{2}\left[+\frac{1}{5\sqrt{5}}\right] = \frac{4}{5} + \frac{1}{5\sqrt{5}}\log\frac{3-\sqrt{5}}{3+\sqrt{5}}$$
$$= \frac{4}{5} - \frac{1}{5\sqrt{5}}\log\frac{2}{7-3\sqrt{5}} = \frac{4}{5} - \frac{1}{5\sqrt{5}}\log\frac{14+6\sqrt{5}}{4} = \frac{4}{5} - \frac{4}{5\sqrt{5}}\log\frac{1+\sqrt{5}}{2}.$$

This is the claim and we may conclude.

Second identity 152

We claim that

$$\frac{\arcsin x}{x\sqrt{1-x^2}} = \sum_{n\ge 0} 4^n \frac{1}{2n+1} \binom{2n}{n}^{-1} x^{2n}.$$

For starters we have

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} = \sum_{n\ge 0} \frac{1}{4^n}\binom{2n}{n} x^{2n}.$$

Hence by integrating we obtain

$$\arcsin x = \sum_{n\ge 0} \frac{1}{4^n}\binom{2n}{n}\frac{1}{2n+1} x^{2n+1}.$$

The coefficient on odd powers of the LHS is zero. The one for even powers is

$$\begin{aligned}
[x^{2n}]\frac{\arcsin x}{x\sqrt{1-x^2}} &= [x^{2n}]\sum_{k\ge 0}\frac{1}{4^k}\binom{2k}{k}\frac{1}{2k+1}x^{2k} \times \sum_{m\ge 0}\frac{1}{4^m}\binom{2m}{m}x^{2m} \\
&= [x^n]\sum_{k\ge 0}\frac{1}{4^k}\binom{2k}{k}\frac{1}{2k+1}x^{k} \times \sum_{m\ge 0}\frac{1}{4^m}\binom{2m}{m}x^{m} \\
&= \sum_{k=0}^{n}\frac{1}{4^k}\binom{2k}{k}\frac{1}{2k+1}\frac{1}{4^{n-k}}\binom{2n-2k}{n-k}.
\end{aligned}$$

We see that this is equivalent to showing

$$\frac{4^{2n}}{2n+1}\binom{2n}{n}^{-1} = \sum_{k=0}^{n}\binom{2k}{k}\frac{1}{2k+1}\binom{2n-2k}{n-k}$$

which we did earlier in identity 166. Multiplying by x and differentiating we get

$$\frac{1}{1-x^2} + \frac{x\arcsin x}{(1-x^2)^{3/2}} = \sum_{n\ge 0} 4^n \binom{2n}{n}^{-1} x^{2n}.$$

Setting $x = i/2$ we find

$$\begin{aligned}
\frac{4}{5} + \frac{1}{2} i \arcsin(i/2)\frac{8}{5\sqrt{5}} &= \frac{4}{5} - \frac{1}{2}\log\left(\sqrt{\frac{5}{4}} + 1/2\right)\frac{8}{5\sqrt{5}} \\
&= \frac{4}{5} - \log\frac{1+\sqrt{5}}{2}\frac{4}{5\sqrt{5}}
\end{aligned}$$

confirming the complex variable argument.

The inverse binomial coefficient identity is from the paper [Mar25].

References

[Mar25] Marko Riedel, Markus Scheuer, and Hosam Mahmoud. Inverse binomial coefficients in egorychev method. *OJAC*, 2025.

[Spi19] M. Spivey. *The Art of Proving Binomial Identities.* CRC Press, 2019.