

Set partitions into labeled tori

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We propose to study a combinatorial class $\mathcal{T} = \text{SET}(\text{TOR}_{\geq 1}(\mathcal{Z}))$ which are partitions of $[n]$ into sets of tori. Just as permutations are set partitions into cycles with class $\text{SET}(\text{CYC}_{\geq 1}(\mathcal{Z}))$ and counted by Stirling cycle numbers we investigate the next dimension, where cycles become tori. Here tori are obtained by attaching opposite sides of a grid with some number of rows and columns, into which the labels are distributed. Toroidal symmetry arises from the rotations of the torus, it can rotate horizontally around the central axis perpendicular to the plane where it rests or vertically about the circle passing through the path traced by the center of the circle that generates the torus by rotating around the central axis. This is topological rather than metric. The symmetries are those pairs of the two types of rotation that take some slot to some other slot.

We introduce a combinatorial class as in *Analytic Combinatorics* by P. Flajolet as follows:

$$\text{TOR}_{=k}(\mathcal{Z})$$

which counts tori on k nodes. Set partitions into labeled tori are then given by

$$\text{SET}(\mathcal{U} \times \text{TOR}_{=1}(\mathcal{Z}) + \mathcal{U} \times \text{TOR}_{=2}(\mathcal{Z}) + \mathcal{U} \times \text{TOR}_{=3}(\mathcal{Z}) + \dots).$$

Now we ask about the generating function of the tori on k nodes. First, if they are to form a grid, that source grid with the attached opposite sides must have $d \times k/d$ rows and columns so that we get a total of k nodes, here d is a divisor of k . Furthermore the group of the symmetries on the k slots has $d \times k/d$ elements. Hence the EGF is

$$Q_k(z) = \frac{z^k}{k!} \sum_{d|k} \frac{k!}{d \times k/d} = \frac{\tau(k)}{k} z^k.$$

where $Q_0(z) = 1$ and $\tau(k)$ is the familiar count of divisors function from number theory. The class of set partitions into labeled tori can then be written as

$$\text{SET}(\mathcal{U} \times \text{TOR}_{\geq 1}(\mathcal{Z}))$$

with EGF

$$Q(z, u) = \exp\left(u \sum_{k \geq 1} \frac{\tau(k)}{k} z^k\right).$$

Compare with $\exp\left(u \left(\log \frac{1}{1-z}\right)\right)$ for Stirling cycle numbers. We set the upper range on k to n when employing this function. We will present a recurrence however in order to be able to effectively work with these quantities.

The coefficient $n![z^n]Q(z, 1)$ generates the sequence of partitions into labeled tori:

$$1, 3, 11, 59, 339, 2629, 20677, 202089, \\ 2066201, 24322931, \dots$$

This is tabulated at [OEIS A028342](#). As an example the coefficient $F_n(u) = n![z^n]Q(z, u)$ gives the OGF of these partitions on n slots by the number of tori that appear. For example with $n = 7$ we get

$$7!F_7(u) = u^7 + 42u^6 + 560u^5 + 3150u^4 + 7588u^3 + 7896u^2 + 1440u.$$

Computing a recurrence

With the goal being to compute the coefficient $\rho_{n,k} = [z^n][u^k]Q(z, u)$ we can extract the coefficient on $[u^k]$ by inspection and are left with

$$\begin{aligned} \rho_{n,k} &= [z^n] \frac{1}{k!} \left(\sum_{m \geq 1} \frac{\tau(m)}{m} z^m \right)^k \\ &= \frac{1}{n} [z^{n-1}] \frac{1}{(k-1)!} \left(\sum_{m \geq 1} \frac{\tau(m)}{m} z^m \right)^{k-1} \sum_{m \geq 1} \tau(m) z^{m-1} \\ &= \frac{1}{n} \sum_{q=0}^{n-1} \rho_{n-1-q, k-1} [z^q] \sum_{m \geq 1} \tau(m) z^{m-1} \\ &= \frac{1}{n} \sum_{q=0}^{n-1} \rho_{n-1-q, k-1} \tau(q+1). \end{aligned}$$

The base case here is $\rho_{n,0} = \delta_{n,0}$. With this recurrence we can easily compute these values even for large n and the initial segment of the data yields the following triangular array for $n \geq 1$:

1									
2	1								
4	6	1							
18	28	12	1						
48	170	100	20	1					
480	988	870	260	30	1				
1440	7896	7588	3150	560	42	1			
20160	60492	73808	37408	9100	1064	56	1		
120960	555264	764524	460656	140448	22428	1848	72	1	
1451520	5819904	8448120	5952700	2162160	436296	49140	3000	90	1

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Note that the first column is tabulated at [OEIS A318249](#) which has $(n-1)! \tau(n)$. This is correct indeed as here we have only one torus with EGF $\tau(n)z^n/n$ and on multiplying by $n!$ get the posted closed form. The entire triangular array is documented at [OEIS A338805](#).

A conjecture

We introduce a random variable X giving the number of tori in a random set partition into tori i.e. k and ask about the expectation. We may use the formula

$$E[X] = \frac{\frac{d}{du} [z^n] Q(z, u) \Big|_{u=1}}{[z^n] Q(z, u) \Big|_{u=1}}.$$

The numerics provide evidence that

$$E[X] \sim N \log^2(n)$$

where $N \approx 0.337$. There might be a lower order term in n on the constant. The graph below illustrates this behavior where the data are displayed as what the recurrence produces.

