Set partitions into labeled 3-tori

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We propose to study a combinatorial class $\mathcal{T} = \text{SET}(\text{TOR}_{3,\geq 1}(\mathcal{Z}))$ which are partitions of $[n]$ into sets of 3-tori. Just as permutations are set partitions into cycles with class $\text{SET}(\text{CYC}_{\geq 1}(\mathcal{Z}))$ and counted by Stirling cycle numbers we investigate the next to next dimension, where cycles become tori and tori become 3-tori. A 3-torus is the four-space analogue of a cycle (two-space) or an ordinary torus (ring, three-space). Here 3-tori are obtained by attaching opposite faces of a rectangular cuboid to each other, with some number $d \times e$ of rows and columns and some height f, into which the labels are distributed. The cuboid does not have slots only on its sides, it is filled with constituent slots, of which there are $d \times e \times f$. Sides are attached so that the slots on the faces become adjacent to the opposite slot on the opposite face. Just as with cycles and ordinary tori the toroidal symmetry arises from the rotations of the 3-torus, the rotations are parallel to the three axes in space and symmetries are those tuples of the three types of rotation that take some slot to some other slot. When we rotate in some direction we must imagine the far face being placed in front of the near one. For example, a vertical rotation shifts everything up by one unit and moves the top face to the bottom.

We introduce a combinatorial class as in *Analytic Combinatorics* by P. Flajolet and R. Sedgewick as follows:

$$\text{TOR}_{3,=k}(\mathcal{Z})$$

which counts 3-tori on k nodes. Set partitions into labeled tori are then given by

$$\text{SET}(\mathcal{U} \times \text{TOR}_{3,=1}(\mathcal{Z}) + \mathcal{U} \times \text{TOR}_{3,=2}(\mathcal{Z}) + \mathcal{U} \times \text{TOR}_{3,=3}(\mathcal{Z}) + \cdots).$$

Now we ask about the generating function of the 3-tori on k nodes. First, if they are to form a cuboid, that cuboid with the attached opposite sides must have d rows and and e columns and f vertical layers (height) so that we get a total of k nodes, here $d \times e \times f = k$. Furthermore the group of the symmetries on the k slots has $d \times e \times f$ elements. Hence the EGF is

$$Q_k(z) = \frac{z^k}{k!} \sum_{def=k} \frac{k!}{d \times e \times f} = \frac{\tau_3(k)}{k} z^k.$$

where $Q_0(z) = 1$ and $\tau_3(k)$ is the function from number theory that counts ordered factorizations into three factors with Dirichlet generating function $\zeta(s)^3$ and closed form for $k = \prod_p p^v$ being $\prod_p \binom{v+2}{2}$ by stars-and-bars. The class of set partitions into labeled tori can then be written as

$$\text{SET}(\mathcal{U} \times \text{TOR}_{3,\geq 1}(\mathcal{Z}))$$

with EGF

$$Q(z, u) = \exp\left(u \sum_{k \geq 1} \frac{\tau_3(k)}{k} z^k \right).$$

Compare with $\exp\left(u \left(\log \frac{1}{1-z} \right) \right)$ for Stirling cycle numbers. We set the upper range on k to n when employing this function. We will present a recurrence however in order to be able to effectively work with these quantities.

The coefficient $n![z^n]Q(z, 1)$ generates the sequence of partitions into labeled 3-tori:

$$1, 4, 16, 106, 658, 6088, 51952, 592828, 6577948, 88213744,$$
$$1173121024, 18663391096, 289030343704,$$
$$5157010548064, 92428084599232, \ldots$$

This is tabulated at [OEIS A318695](). As an example the coefficient $F_n(u) = n![z^n]Q(z, u)$ gives the OGF of these partitions on n slots by the number of 3-tori that appear. For example with $n = 7$ we get

$$F_7(u) = u^7 + 63u^6 + 1155u^5 + 7875u^4 + 21042u^3 + 19656u^2 + 2160u.$$

Computing a recurrence

With the goal being to compute the coefficient $\rho_{n,k} = [z^n][u^k]Q(z,u)$ we can extract the coefficient on $[u^k]$ by inspection and are left with

$$\rho_{n,k} = [z^n]\frac{1}{k!}\left(\sum_{m\geq 1}\frac{\tau_3(m)}{m}z^m\right)^k$$

$$= \frac{1}{n}[z^{n-1}]\frac{1}{(k-1)!}\left(\sum_{m\geq 1}\frac{\tau_3(m)}{m}z^m\right)^{k-1}\sum_{m\geq 1}\tau_3(m)z^{m-1}$$

$$= \frac{1}{n}\sum_{q=0}^{n-1}\rho_{n-1-q,k-1}[z^q]\sum_{m\geq 1}\tau_3(m)z^{m-1}$$

$$= \frac{1}{n}\sum_{q=0}^{n-1}\rho_{n-1-q,k-1}\tau_3(q+1).$$

The base case here is $\rho_{n,0} = \delta_{n,0}$. With this recurrence we can easily compute these values even for large n and the initial segment of the data yields the following triangular array for $n \geq 1$:

1									
3	1								
6	9	1							
36	51	18	1						
72	360	195	30	1					
1080	2412	2025	525	45	1				
2160	19656	21042	7875	1155	63	1			
50400	177552	225288	113337	23940	2226	84	1		
241920	1557792	2642112	1614816	456057	61236	3906	108	1	
3265920	19596384	31544640	23793480	8371755	1497069	137970	6390	135	1

Note that the first column is $(n-1)!\tau_3(n)$. This is correct indeed as here we have only one 3-torus with EGF $\tau_3(n)z^n/n$ and on multiplying by $n!$ get the matching form. Note furthermore that on the next-to-last diagonal we have partitions of $[n]$ into $n-1$ 3-tori, so there is one 3-torus of dimension $1 \times 1 \times 2$, or $1 \times 2 \times 1$ or $2 \times 1 \times 1$ for which we must choose two labels with the rest being singletons and we get $3\binom{n}{2}$, which is tabulated as matchstick numbers under OEIS A045943. The rotation along the axis containing the two slots forms a total of one orbit.

A conjecture

We introduce a random variable X giving the number of 3-tori in a random set partition into 3-tori i.e. k and ask about the expectation. We may use the formula

$$\mathrm{E}[X] = \frac{\frac{d}{du}[z^n]Q(z,u)\big|_{u=1}}{[z^n]Q(z,u)\big|_{u=1}}.$$

The numerics provide evidence that

$$\boxed{\mathrm{E}[X] \sim N\log^3(n)}$$

where $N \approx 0.08.$ There might be a lower order term in n on the constant. The graph below illustrates this behavior where the data are displayed as what the recurrence produces.