Egorychev Method: A hidden treasure

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Egorychev

Dedication to Egorychev

Winner of Fulkerson Prize for proving van der Waerden's conjecture.
Professor at Siberian Federal University
Egorychev, G. (1984). Integral Representation and the Computation of Combinatorial Sums. American Mathematical Society. Providence, Rhode Island.

Marko Riedel

Riedel, M. (2022). *Egorychev Method*. Wikipedia entry.
Riedel, M. (2022). *Egorychev Method and the Evaluation of Combinatorial Sums*. Internet source, published by the author.
Programmer in various languages.

Does mathematical research in his spare time.

Experience in system administrator and considers himself a Perl enthusiast.

Outline

- The talk is intended for a general audience (students, mathematicians, economists, engineers, scientists, statisticians, cardiologists).
- Scope: What is Egorychev method all about?
- A crash course in functions of complex variables.
- Outline of Egorychev method: three flavors, I, II, III.
- Example of Egorychev I.
- Example of Egorychev III.
- Residue calculus, and two examples of Egorychev II.
- Time permitting: one example done by all three variations.
- Time permitting: more examples.

Scope: Schläfli's identity

Is there anyway to reduce the sum?

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n+j-1}{n-k-1}\binom{n+k}{k-j}\left\{{j+k \atop j}\right\}.$$

Where do we start? Egorychev method is algorithm-like: steps if you follow will give you the reduction:

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n+j-1}{n-k-1}\binom{n+k}{k-j}\left\{{j+k \atop j}\right\} = \left[{n \atop n-k}\right].$$

$\left[{n \atop k}\right]$:= signless Stirling numbers of the first kind.

$\left\{{n \atop k}\right\}$:= Stirling numbers of the second kind.

Complex functions

Some basic knowledge of complex variable theory is assumed, but not a whole lot:

$$i = \sqrt{-1}\,.$$

We use the form $z = x + iy$, for complex variable. The symbol $|z|$ stands for the *magnitude* $\sqrt{x^2 + y^2}$ of z.
We consider a function (such as z^2 or e^z) as a *function of a complex variable*, **complex function**.

Derivatives

A complex function derivative f, can have a derivative, defined in the usual way as the rate of change (possibly complex) of f at z,

$$\frac{f(z+h)-f(z)}{h}$$

for a displacement h (in infinitesimal complex units) , as such displacement tends to 0 in magnitude.

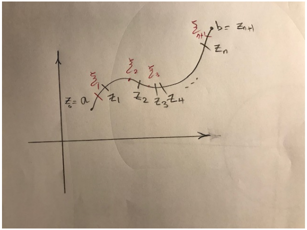
To guarantee the existence of such a derivative, regardless of the path of approach, Cauchy-Riemann equations are necessary and sufficient conditions.

It is not our intention to get into these conditions as line integrals are more relevant to our purpose.

Line integrals

Formally, line integration is defined as a limit of approximations.

Suppose L is a line (not necessarily straight) joining the two points a and b in the z-complex plane. Divide the line L into segments by choosing n distinct points on it; call the dividing points $z_1, \dots, z_n$. Think of a as z_0 and of b as z_{n+1}. From each of the $n+1$ segments choose a point. The point from the kth segment is to be called ξ_k.



Line integrals

Consider the (complex) sum

$$S_n = \sum_{k=1}^{n+1} f(\xi_k)\,\nabla z_k,$$

where ∇z_k is the backward difference $z_k - z_{k-1}$. Let $n \to \infty$ in such a way that the largest absolute difference $|\nabla z_k|$ converges to 0, i.e., $\max_{1\leq k\leq n} |\nabla z_k| \to 0$. If S_n converges to a limit, we call that limit the integral of $f(z)$ on L and denote it by $\int_L f(z)\,dz$. When the line L is not a self-intersecting closed curve, forming a contour, we write the contour integration in the notation

$$\oint_L f(z)\,dz.$$

Generating functions versus formal power series

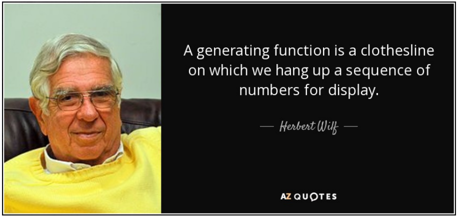
Generating functions are a way to represent sequences of numbers (real or complex). Suppose $a_0, a_1, a_2, \ldots$ is a given sequence of complex numbers. The representation

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

is called an *ordinary generating function* of the sequence $\{a_n\}_{n=0}^{\infty}$. One thinks of the powers of z as placeholders of the coefficients: z^n has the coefficient a_n.

Laundry line

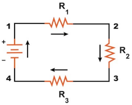
Herbert Wilf, Generatingfunctionology Academic Press, (1990).
Famous quote



Formal power series

Such a representation is very useful in itself as a formal power series.

The expression *formal power series* stands for a series of powers without regard to the possibility of convergence.



But, it often aids a derivation, when the representation appears in closed form or coincides with a known elementary function.

Exponential generating functions

Sometimes, it is more convenient to work with the ordinary generating function of the sequence $\{a_n/n!\}_{n=0}^{\infty}$, which is

$$g(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}.$$

In this form, the generating function is called the *exponential generating function* of the sequence $\{a_n\}_{n=0}^{\infty}$.
When valuation is desired, the presence of the fast-growing factorials in the exponential generating function helps the convergence of the exponential generating function toward a limit.

Example

$$a_n = 1.$$

Such a sequence has ordinary generating function

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

As a formal power series, z^n holds the coefficient 1 in front of it. However, this geometric series is convergent only within the disc $|z| < 1$, and when it converges, it approaches $h(z) = 1/(1-z)$.

$$f\left(\frac{1}{4}\right) = \frac{4}{3},$$

$f(7)$ has no meaning (other than the undefined ∞), though $h(7) = -1/6$.
$h(z)$ is the *analytic continuation* of $f(z)$, a function that agrees with $f(z)$, wherever $f(z)$ is defined, but extends the definition to a domain in the z-complex plane that is larger than that of $f(z)$.

Convergence issues

In contrast, the sequence of 1's has the exponential generating function

$$g(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z,$$

well defined and convergent at every point in the z-complex plane.

The Extractor

We use the notation $[z^n]\, h(z)$ to extract the nth coefficient in the formal power series expansion of $h(z)$. Thus, if $f(z)$ is the ordinary generating function of the sequence $\{a_n\}_{n=0}^{\infty}$, then

$$[z^n]\, f(z) = a_n.$$

For instance, for the sequence of 1's

$$[z^n]\, f(z) = 1, \qquad [z^n]\, g(z) = \frac{1}{n!}.$$

Popularized in Philippe Flajolet's work.
Alden Biesen (Belgium), 2006.

Shifting and scaling

Shifting

$$[z^{n-k}]\, f(z) = [z^n]\, z^k f(z),$$

$$z^k f(z) = a_0 z^k + a_1 z^{k+1} + \cdots + a_{n-k} z^n + \cdots \ .$$

To find the nth coefficient a_n, look for the power of $n-k$ in $f(z)$.

Rescaling

$$q^n [z^n]\, f(z) = [z^n]\, f(qz),$$

both sides are $q^n a_n$.
Power shifting is magic. Formulated and popularized in Riedel's work.

The algebra of formal power series

When a generating function does not converge, it may still be viewed as a useful formal power series.
For instance, the expression

$$f(z) = \sum_{n=0}^{\infty} n!\, z^n$$

is a formal power series for the factorials, though it is not convergent for any z in the complex plane, except at $z = 0$.
With the notion of convergence set aside, it is easier to build an "algebra" for formal power series than for convergent series, since the conditions are relaxed.

The algebra of formal power series

In essence, the algebra on formal power series is an algebra on the sequences they represent.
Suppose $f_1(z) = \sum_{n=0}^{\infty} a_n\, z^n$ and $f_2(z) = \sum_{n=0}^{\infty} b_n\, z^n$ are two formal power series representing the two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$.
We define the sum

$$f_1(z) + f_2(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n,$$

without any regard to the convergence of either formal power series or the outcome of the addition.

Sums and differences

For instance, if

$$f_1(z) = \sum_{n=0}^{\infty} z^n, \qquad f_2(z) = \sum_{n=0}^{\infty} n!\, z^n,$$

we accept their sum as

$$f_1(z) + f_2(z) = \sum_{n=0}^{\infty} (1 + n!) z^n,$$

generating the sequence $1 + n!$, even though $f_2(z)$ is not convergent anywhere, except at $z = 0$ (nor is the sum).
Local focus (convergence is a global property).
Similarly, we define the difference

$$f_1(z) - f_2(z) = \sum_{n=0}^{\infty} (a_n - b_n) z^n,$$

Products

$$f_1(z)f_2(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} z^n.$$

Note that in the definition of the product, only a finite portion of both sequences is engaged, emphasizing the “local” aspect of the operation at each index n.

Analytic functions

A function f of the complex variable z is called *analytic* at z, if it is differentiable in an open neighborhood of z.

Example

$$f(z) = e^z.$$

is analytic at $z = 3$, in fact everywhere.

The function

$$f(z) = \frac{2}{z-3}.$$

is not analytic at 3. In fact, its analytic everywhere, except at $z = 3$

Singularities

If $f(z)$ is not analytic at z, the point z is considered a *singularity*, a point where a complex function is not well behaved (non-differentiable, infinite, for example).

Poles

If there is $n \geq 1$, such that $(z - z_0)^n f(z)$ has a nonzero limit as z approaches z_0, we say z_0 is a *pole* of $f(z)$ of order n.

Example

$$f(z) = \frac{2z}{(z-4)(z-(2-3i))^5}.$$

The points 4 and $2 - 3i$ are poles, of orders 1 and 5.

Residues

The residue of a function f with a pole of order m at z_0 is

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z).$$

The residue at infinity has a special definition:

$$\operatorname*{res}_{z=\infty} f(z) = -\frac{1}{z^2} \operatorname*{res}_{z} f\left(\frac{1}{z}\right).$$

For instance, we have

$$\operatorname*{res}_{z=0} \frac{3}{z(z+1)} = \lim_{z\to 0} z \times \frac{3}{z(z+1)} = 3;$$
$$\operatorname*{res}_{z=-1} \frac{3}{z(z+1)} = \lim_{z\to -1}(z+1) \times \frac{3}{z(z+1)} = -3,$$

and

$$\operatorname*{res}_{z=2} \frac{5}{z(z-2)^2} = \lim_{z\to 2} \frac{d}{dz}(z-2)^2 \times \frac{5}{z(z-2)^2} = -\frac{5}{4}.$$

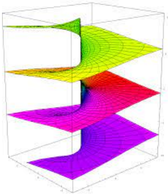
Branch points

A point for which one turn on **any** closed path around it does not bring the function back to the same value.

Example

$$\ln z = \ln e^{2k\pi i} = 2\pi i k, \qquad \text{for} k = 0, 1, 2, \ldots \ .$$

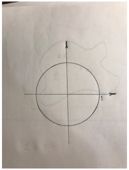
After one turn $\ln(1) = 0$;
after two turns $\ln(1) = 2\pi i$.
You go up one floor on the so-called Riemann surfaces:



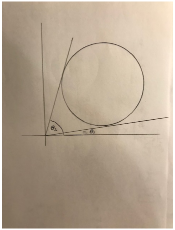
Branch points

For $\ln z$

The origin (0,0) is a branching point:



The point (1,1) is not a branching point:

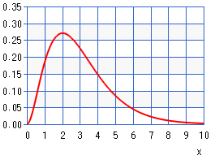


Branch cuts

A line if crossed while turning, the function changes its value.

Not unique.

A (possible) branch cut for $\ln z$.



Meromorphic functions

A function $f(z)$ is *meromorphic* in a region, if it is analytic in that region, except for a finite number of poles. This is mostly what appears in applications of Egorychev method.

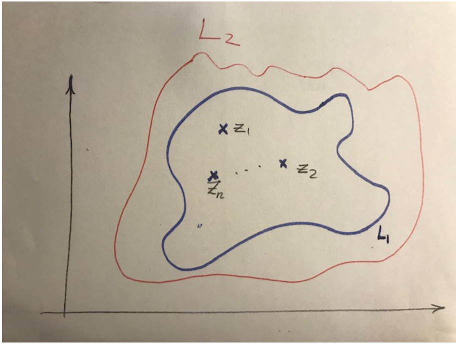
Even when we make a transformation that introduces a branching points, we craft contours avoiding them and not crossing their branch cuts.

Other types of singularities

Removable singularity: $(z-1)/(z^2-1)$, easy to recognize and deal with.

Essential singularity: If $f(z)$ is not differentiable at z and the singularity is not one of the other three types, it is called an *essential singularity*. As an example, the function $e^{1/z}$ has an essential singularity at 0.

Cauchy's residue theorem



$$\oint_L f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z).$$

$$\oint_L \frac{dz}{z^2} = 0, \qquad \oint_L \frac{dz}{z} = 2\pi i, \qquad \oint_L z^2\,dz = 0.$$

The nitty-gritty: Egorychev method

In his book, Egorychev outlines a potent method for the reduction of combinatorial sums.

The chief idea is to identify parts that can be summed in closed form, then treat the leftover via series (formal power series, residues, Cauchy integrals).

How do we identify summable part: EgorychevI, II, III

There are three variations; the chief idea in all three nuances is to identify parts that can be summed in closed form by:

(I) recognizing by direct inspection a combinatorial connection of certain coefficients in the summand to the coefficients in the expansion of a known formal power series,

(II) by replacing some factors in the summand with residue operators,

(III) or by replacing some factors in the summand with Cauchy contour integrals as i, for example.

What then?

Once the summation is gone, what is left is a *core function* (under an operator like an extractor, a residue or a Cauchy integral. From the core the extraction of coefficients is done by direct inspection, by computing residues, or by evaluating an integral via Cauchy's integral formula.

Egorychev I: The most direct, recognition of standard formal power series.
Egorychev II: convenient, when the coefficients are residues of functions (Egorychev's algorithmic approach).
Egorychev III: is actual integration. This heavy machinery should be reserved as a last resort for cases, where the core has poles of arbitrarily high orders inside and outside *any* chosen contour.

Binomial coefficients

Before I start any examples, we are using the analytic continuation of the usual binomial coefficients

$$\binom{z}{k} = \frac{z(z-1)\dots(z-k+1)}{k!}.$$

$$\binom{8}{3}, \qquad \binom{1/2}{3}, \qquad \binom{-1/2}{3} \qquad \binom{5+2i}{3}$$

all have meanings.

For example

$$\binom{-1/2}{3} = \frac{(-1/2)(-3/2)(-5/2)}{6} = -\frac{5}{16}.$$

Example: Egorychev I

$$\sum_{k=0}^{n}(-4)^k\binom{p}{k}\binom{2p-2k}{n-k} = (-1)^n\binom{2p}{n},$$

By the power shift technique, we have

$$\begin{aligned} S_n &= \sum_{k=0}^{n}(-4)^k\binom{p}{k}[z^{n-k}](1+z)^{2p-2k} \\ &= \sum_{k=0}^{n}(-4)^k\binom{p}{k}[z^n]\,z^k(1+z)^{2p-2k} \\ &= [z^n](1+z)^{2p}\sum_{k=0}^{\infty}\binom{p}{k}\frac{(-4z)^k}{(1+z)^{2k}}. \end{aligned}$$

We have identified a sum that has a closed form (by the binomial theorem):

Let's get a core

$$
\begin{aligned}
S_n &= [z^n]\,(1+z)^{2p} \sum_{k=0}^{\infty} \binom{p}{k} \frac{(-4z)^k}{(1+z)^{2k}} \\
&[z^n]\,(1+z)^{2p} \left(1 - \frac{4z}{(1+z)^2}\right)^p \\
&= [z^n]\,\left((1+z)^2 - 4z\right)^p \\
&= [z^n]\,(z^2 - 2z + 1)^p \\
&= [z^n]\,(1-z)^{2p} \\
&= (-1)^n \binom{2p}{n}.
\end{aligned}
$$

Gould's extension

This is Example 3 in Egorychev's book.
The result then holds for any complex p, because both sides are polynomials in z of degree n. Once two polynomials of degree n agree at n points, they are the same everywhere in the complex plane. This is a technique used frequently in Gould's compilation (an Internet source, self published by the author).

$$\sum_{k=0}^{n}(-4)^k\binom{z}{k}\binom{2z-2k}{n-k}=(-1)^n\binom{2z}{n},$$

for any complex z.

Example: Egorychev III

We establish the identity

$$\sum_{\substack{k=0 \\ k \text{ odd}}}^{m} \binom{2n}{k}\binom{2m-2n}{m-k} = \frac{1}{2}\binom{2m}{m} + (-4)^m \binom{n-1/2}{m}.$$

This is Problem 3782050 on math.stackexchange.com. Toward a proof, let us call the sum $S_{n,m}$. To avoid skipping indices, we work with the difference between two full sums:

$$2S_{n,m} = S_{n,m}^{(1)} - S_{n,m}^{(2)},$$

where

$$S_{n,m}^{(1)} := \sum_{k=0}^{m} \binom{2n}{k}\binom{2m-2n}{m-k} = \binom{2m}{m};$$

$$S_{n,m}^{(2)} := \sum_{k=0}^{m} (-1)^k \binom{2n}{k}\binom{2m-2n}{m-k}.$$

$S_{n,m}^{(1)}$

$$S_{n,m}^{(1)} := \sum_{k=0}^{m} \binom{2n}{k} \binom{2m-2n}{m-k} = \binom{2m}{m}$$

In $S_{n,m}^{(1)}$ we can use Vandermonde identity, which has an algebraic proof, a combinatorial proof, and a probability proof. We point out that it can also be proved straightforwardly via Egorychev I.

$S_{n,m}^{(2)}$

We re-index the sum $S_{n,m}^{(2)}$ by setting $j = m - k$ and switch the sign of the upper index of one binomial coefficient

$$
\begin{aligned}
S_{n,m}^{(2)} &:= \sum_{k=0}^{m} (-1)^k \binom{2n}{k} \binom{2m-2n}{m-k} \\
&= (-1)^m \sum_{j=0}^{\infty} (-1)^j \binom{2n}{2n-m+j} \binom{2m-2n}{j} \\
&= (-1)^m \sum_{j=0}^{\infty} (-1)^j (-1)^{2n-m+j} \binom{-m+j-1}{2n-m+j} \binom{2m-2n}{j} \\
&= (-1)^m \sum_{j=0}^{\infty} (-1)^j \binom{2m-2n}{j} \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{dz}{z^{2n-m+j+1}(1-z)^{m-j+1}}.
\end{aligned}
$$

$$S_{n,m}^{(2)} = \frac{(-1)^m}{2\pi i} \oint_{|z|=\varepsilon} \frac{1}{z^{2n-m+1}(1-z)^{m+1}} \times \sum_{j=0}^{\infty} (-1)^j \binom{2m-2n}{j} \frac{(1-z)^j}{z^j} \, dz,$$

for some $\varepsilon \in (0,1)$. We shall specify a suitable value for ε after a transformation.

We have identified a part that can be summed in closed form by the binomial theorem.

The core

$$
\begin{aligned}
S^{(2)}_{n,m} &= \frac{(-1)^m}{2\pi i} \oint_{|z|=\varepsilon} \frac{1}{z^{2n-m+1}(1-z)^{m+1}} \left(1 - \frac{1-z}{z}\right)^{2m-2n} dz \\
&= \frac{(-1)^m}{2\pi i} \oint_{|z|=\varepsilon} \frac{(1-2z)^{2m+2-2-2n}}{z^{m+1}(1-z)^{m+1}} \, dz.
\end{aligned}
$$

As m is arbitrary, the pole at 0—inside the contour—is of arbitrarily high order. It does not help to try evaluating the integral via the pole at 1—lying outside the contour—which is of arbitrarily high order, too.

A change of integration variable

Put $w = z(1-z)/(1-2z)^2$. Note right away that a big chunk of the integrand is simplified via this transformation. Namely, we have $(1-2z)^2/(z(1-z)) = 1/w$, and the part $(1-2z)^{2m+2}/z^{m+1}(1-z)^{m+1}$ is to be replaced with $1/w^{m+1}$.

Inversion

To invert this relation, to get z in terms of w, we need to solve the quadratic equation

$$(1+4w)z^2 - (1+4w)z + w = 0.$$

The two solutions are

$$z = \frac{1}{2} \pm \frac{1}{2\sqrt{1+4w}}.$$

Of the two branches, we choose the one with the negative sign to have $w = 0$ as the image of $z = 0$.
As we transform the integral, we get a Jacobian:
$dz = dw/(1+4w)^{3/2}$. We have $1 - 2z = (1+4w)^{-\frac{1}{2}}$,

Beware of the branch point

The integrand in the new domain is $w^{-(m+1)}(1+4w)^{n-\frac{1}{2}}$. At $-1/4$ we have a branching point. One possible branch cut in the w-plane is the line $(-\infty, -1/4]$. The function $(1+4w)^{n-\frac{1}{2}}$ is analytic inside the disc $|w| < 1/4$.

Instantiating the contour

Recall $w = z(1-z)/(1-2z)^2$. and z is rotating on a circle of $0 < \varepsilon < 1$
Bounding the magnitude:

$$\frac{\varepsilon(1-\varepsilon)}{(1+2\varepsilon)^2} \leq |w| \leq \frac{\varepsilon(1+\varepsilon)}{(1-2\varepsilon)^2}.$$

However, the transformed contour must not intersect the branch cut, i.e., we must take $\varepsilon(1+\varepsilon)/(1-2\varepsilon)^2 < 1/4$, which says $\varepsilon < 1/8$. We may instantiate ε to $1/9$ to have

$$\frac{8}{121} \leq |w| \leq \frac{10}{49}.$$

The left plot in Figure 1 shows the transformed contour (in red) enclaved in an annulus of outer radius $10/49$ (the green circle) and inner radius $8/121$ (the blue circle).

We get the transformed integral

$$S_{n,m}^{(2)} = \frac{(-1)^m}{2\pi i} \oint_{|w|=8/121} \frac{(1+4w)^{n-\frac{1}{2}}}{w^{m+1}}\, dw = (-4)^m \binom{n-\frac{1}{2}}{m}.$$

The ϵ-γ technique used in the example is general and works well for other more complicated examples, as it did here. It is remarkable that the original contour is a circle, and its image is almost a circle of the same radius.

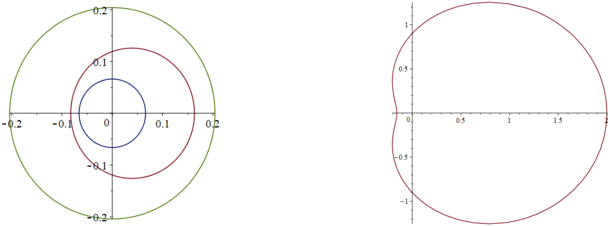


Figure: The transformed contours in the w-plane: (a) left (red), with $|z| = 1/9$; (b) right (red), with $|z| = 1/3$.

Egorychev II: Residue calculus

A certain calculus does apply to residues and renders Egorychev method algorithm-like with certain intermediate steps almost automated.

In his book, Egorychev presents a set of "rules," for finding the residues in multivariate functions of complex variables, such as $f(z_1, \ldots, z_n)$. He then moves to the simplified univariate versions. That's all we need for our examples.

Rules

(i) Removal of the residue operator: The residues $\operatorname*{res}_w A(w)/w^{k+1}$ and $\operatorname*{res}_w B(w)/w^{k+1}$ are equal, for each $k \geq 0$, if and only if $A(w) = B(w)$.

(ii) Linearity: For any complex numbers α and β, we have

$$\operatorname*{res}_w \left(\alpha \frac{A(w)}{w^{k+1}} + \beta \frac{B(w)}{w^{k+1}} \right) = \alpha \operatorname*{res}_w \frac{A(w)}{w^{k+1}} + \beta \operatorname*{res}_w \frac{B(w)}{w^{k+1}},$$

for each $k \geq 0$.

(iii) Substitution of coefficients: For a function $A(z)$ that does not have a pole at zero, we have the expansion

$$A(z) = \sum_{k=0}^{\infty} z^k \operatorname*{res}_w \frac{A(w)}{w^{k+1}}.$$

Rules

(iv) Inversion: Suppose $h(w) = w/f(w)$, and $g(z)$ is the inverse of the series $z = h(w)$, i.e., $g(z) = w$. Suppose further that the free term (coefficient of z^0) in the formal power series $f(z)$ is not equal to zero. For a meromorphic function $A(w)$ that does not have a pole at 0, we have

$$\sum_{k=0}^{\infty} z^k \operatorname*{res}_{w=0} \frac{A(w)\, f^k(w)}{w^{k+1}} = \left(\frac{A\big(g(z)\big)}{f\big(g(z)\big)\, h'\big(g(z)\big)}\right).$$

(v) Change of variable under the residue operator: Suppose $f(z)$ has a nonzero free term, and $h(w) = w/f(w)$. For $w = g(z)$, the inverse of $z = h(w)$, and a meromorphic function $A(w)$ that does not have a pole at 0, we have

$$\operatorname*{res}_{w} \frac{A(w)\, f^k(w)}{w^{k+1}} = \operatorname*{res}_{z} \frac{A\big(g(z)\big)}{z^{k+1} f\big(g(z)\big)\, h'\big(g(z)\big)},$$

for each $k \geq 0$.

Rules

(vi) Differentiation:

$$\operatorname*{res}_{w} \frac{A'(w)}{w^k} = k \operatorname*{res}_{w} \frac{A(w)}{w^{k+1}}, \qquad \text{for each} \quad k \geq 0.$$

(vii) Integration:

$$\frac{1}{k+1} \operatorname*{res}_{w} \frac{A(w)}{w^{k+1}} = \operatorname*{res}_{w} \frac{1}{w^{k+2}} \int_0^w A(s)\, ds, \qquad \text{for each} \quad k \geq 0.$$

Example: Egorychev II

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n+j-1}{n-k-1}\binom{n+k}{k-j}\left\{{j+k \atop j}\right\} = \left[{n \atop n-k}\right].$$

$\left[{n \atop k}\right]$:= signless Stirling numbers of the first kind.

$\left\{{n \atop k}\right\}$:= Stirling numbers of the second kind.

Toward a proof, let us call the sum $S_{n,k}$.

Stirling number of the second kind is a coefficient in a formal power series (exponential generating function) and can be extracted as a residue:

$$\left\{ {n \atop k} \right\} = \frac{n!}{k!} \operatorname*{res}_{z} \frac{(e^z - 1)^k}{z^{n+1}}.$$

We start with re-indexing (setting $j = k - s$), while retrieving the Stirling number as a coefficient of the exponential generating function and using the index shift:

$j = k - s$

$$
\begin{aligned}
S_{n,k} & \sum_{j=0}^{k}(-1)^{k-j}\binom{n+j-1}{n-k-1}\binom{n+k}{k-j}\left\{ {j+k \atop j} \right\} \\
&= \sum_{s=0}^{k}(-1)^{s}\binom{n+k-s-1}{n-k-1}\binom{n+k}{s}\left\{ {2k-s \atop k-s} \right\} \\
&= \sum_{s=0}^{k}(-1)^{s}\frac{(n+k-s-1)!}{(n-k-1)!\,(2k-s)!}\binom{n+k}{s}\frac{(2k-s)!}{(k-s)!}\,[z^{2k-s}]\,(e^{z}-1)^{k-s} \\
&= \frac{(n-1)!}{(n-k-1)!}\sum_{s=0}^{k}(-1)^{s}\frac{(n+k-s-1)!}{(k-s)!\,(n-1)!}\binom{n+k}{s}\,[z^{2k}]\,z^{s}(e^{z}-1)^{k-s} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^{z}-1)^{k}\sum_{s=0}^{k}(-1)^{s}\binom{n+k-s-1}{k-s}\times\binom{n+k}{s}\,z^{s}(e^{z}-1)^{-s}
\end{aligned}
$$

$$
\begin{aligned}
S_{n,k} &= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^k \sum_{s=0}^{k}(-1)^s \binom{n+k-s-1}{k-s} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \binom{n+k}{s} z^s (e^z-1)^{-s} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^k \sum_{s=0}^{k}(-1)^s\,[w^{k-s}]\,(1+w)^{n+k-s-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \binom{n+k}{s} z^s (e^z-1)^{-s} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^k\,[w^k]\,(1+w)^{n+k-1} \\
&\qquad\qquad \times \sum_{s=0}^{k}(-1)^s\, w^s z^s (1+w)^{-s} (e^z-1)^{-s} \times \binom{n+k}{s}.
\end{aligned}
$$

Collecting the terms containing w under the extractor $[w^k]$, we find $(1+w)^{n+k-1}w^s(1+w)^{-s}$, with an expansion starting at at w^s—there is no contribution to the extractor, when $s > k$. Hence, we can extend the sum to go till $n+k$, and identify the binomial theorem expansion:

$$
\begin{aligned}
S_{n,k} &= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^k \\
&\qquad \times [w^k]\,(1+w)^{n+k-1}\left(1-\frac{wz}{(1+w)(e^z-1)}\right)^{n+k} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^{-n} \\
&\qquad \times [w^k]\,(1+w)^{-1}\big(w(e^z-1-z)+(e^z-1)\big)^{n+k} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^{-n}\,[w^k]\,(1-w+w^2-w^3+\cdots) \\
&\qquad \times \sum_{j=0}^{n+k}\binom{n+k}{j}w^j(e^z-1-z)^j(e^z-1)^{n+k-j}.
\end{aligned}
$$

Performing the extraction of the coefficient of w^k, we get

$$
\begin{aligned}
S_{n,k} &= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^{-n} \\
&\qquad \times \sum_{s=0}^{k}(-1)^{k-s}\binom{n+k}{s}(e^z-1-z)^s(e^z-1)^{n+k-s} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^{-n} \\
&\qquad \times \sum_{s=0}^{k}(-1)^{k-s}\binom{n+k}{s}\left(\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)-1-z\right)^s \\
&\qquad\qquad \times \left(\left(1+z+\frac{z^2}{2}+\cdots\right)-1\right)^{n+k-s} \\
&= \frac{(n-1)!}{(n-k-1)!}\,[z^{2k}]\,(e^z-1)^{-n} \\
&\qquad \times \sum_{s=0}^{k}(-1)^{k-s}\binom{n+k}{s}\left(\left(\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)^s\left(z+\frac{z^2}{2}+\cdots\right)^{n+
\end{aligned}
$$

Note that the expansion of $(e^z - 1)^{-n}$ starts at z^{-n}. The summand, if evaluated at $s = k + 1, \ldots, n + k$, would yield expansions starting at powers higher than $n + 2k$. In conjunction with the powers coming from the expansion of $(e^z - 1)^{-n}$ (the least among which is z^{-n}), we get powers of z higher than $2k$. So, we are free to extend s beyond k to $n + k$. The extractor $[z^{2k}]$ annihilates them.

We end up with

$$\begin{aligned} S_{n,k} &= \frac{(n-1)!}{(n-k-1)!} (-1)^k [z^k] \left(\frac{z}{e^z - 1} \right)^n \\ &= \frac{(n-1)!}{(n-k-1)!} (-1)^k \operatorname*{res}_z \frac{1}{z^{k+1}} \left(\frac{z}{e^z - 1} \right)^n . \end{aligned}$$

Now, put $e^z - 1 = v$, so that $z = \ln(1+v)$, with $dz = dv/(1+v)$. Applying a modification of Rule (v); a generalization in Fürst (2001):

$$S_{n,k} = \frac{(n-1)!}{(n-k-1)!}\,(-1)^k \operatorname*{res}_v \frac{1}{v^n} \ln^{n-k-1}(1+v) \times \frac{1}{1+v}.$$

Using the integration rule (Rule (vii)) from the residue calculus, with $A(v) = \frac{1}{v+1}\ln^{n-k-1}(1+v)$, we get

$$\begin{aligned} S_{n,k} &= \frac{(n-1)!}{(n-k-1)!}\,(-1)^k \times n \operatorname*{res}_v \frac{1}{v^{n+1}} \int_0^v \ln^{n-k-1}(1+s)\frac{ds}{s+1} \\ &= \frac{(n-1)!}{(n-k-1)!}\,(-1)^k\,[v^n]\,\frac{n}{n-k} \ln^{n-k}(1+v) \\ &= \frac{n!}{(n-k)!}\,(-1)^{k+n}\,[v^n] \ln^{n-k}(1-v) \end{aligned}$$

$$S_{n,k} = \frac{n!}{(n-k)!}\,(-1)^{k+n}\,[v^n]\,\ln^{n-k}(1-v)$$
$$= \frac{n!}{(n-k)!}\,[v^n]\,\ln^{n-k}\frac{1}{1-v};$$

in the penultimate step, the sign $(-1)^n$ came into the expression by enacting the rescaling property of extractors from a formal power series, with the scale -1.
This is the nth coefficient in the exponential generating function of Stirling numbers of the first kind.

$$S_{n,k} = \left[{n \atop n-k} \right].$$

Comparing the methods

One example, three variations
we establish the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{n}\binom{k}{j}=(-1)^n\binom{n}{j}\binom{n+j}{j}$$

by all three variations.
Let us call the given sum as $S_{n,j}$.

Egorychev I

Here we identify a summable part and a core that happens to have a well-known formal power series; it is simply the binomial theorem:

$$\begin{aligned}
S_{n,j} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} [z^n] (1+z)^{n+k} [w^j] (1+w)^k \\
&= [z^n] (1+z)^n [w^j] \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+z)^k (1+w)^k \\
&= [z^n] (1+z)^n [w^j] \big(1 - (1+z)(1+w)\big)^n \\
&= (-1)^n [z^n] (1+z)^n [w^j] \big(z + w(1+z)\big)^n \\
&= (-1)^n [z^n] (1+z)^n \binom{n}{j} z^{n-j} (1+z)^j \\
&= (-1)^n \binom{n}{j} [z^j] (1+z)^{n+j} \\
&= (-1)^n \binom{n}{j} \binom{n+j}{j}.
\end{aligned}$$

Egorychev II

Here again, we identify a summable part wile extracting coefficients as residues:

$$
\begin{aligned}
S_{n,j} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \operatorname*{res}_z \frac{(1+z)^{n+k}}{z^{n+1}} \operatorname*{res}_w \frac{(1+w)^k}{w^{j+1}} \\
&= \operatorname*{res}_z \frac{(1+z)^n}{z^{n+1}} \operatorname*{res}_w \frac{1}{w^{j+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+z)^k (1+w)^k \\
&= \operatorname*{res}_z \frac{(1+z)^n}{z^{n+1}} \operatorname*{res}_w \frac{1}{w^{j+1}} \left(1-(1+z)(1+w)\right)^n \\
&= (-1)^n \operatorname*{res}_z \frac{(1+z)^n}{z^{n+1}} \operatorname*{res}_w \frac{1}{w^{j+1}} \left(z+w(1+z)\right)^n \\
&= (-1)^n \operatorname*{res}_z \frac{(1+z)^n z^{n-j} (1+z)^j}{z^{n+1}} \binom{n}{j} \\
&= (-1)^n \binom{n}{j} \operatorname*{res}_z \frac{(1+z)^{n+j}}{z^{j+1}} \\
&= (-1)^n \binom{n}{j} \binom{n+j}{j}.
\end{aligned}
$$

Egorychev III

Extract the second and third binomial coefficients by Cauchy integrals:

$$\begin{aligned}
S_{n,j} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{2\pi i} \oint_{|z|=1} \frac{(1+z)^{n+k}}{z^{n+1}} \times \frac{1}{2\pi i} \oint_{|w|=1} \frac{(1+w)^k}{w^{j+1}} \, dw \, dz \\
&= \frac{1}{2\pi i} \oint_{|z|=1} \frac{(1+z)^n}{z^{n+1}} \\
&\qquad \times \frac{1}{2\pi i} \oint_{|w|=1} \frac{1}{w^{j+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+z)^k (1+w)^k \, dw \, dz \\
&= \frac{1}{2\pi i} \oint_{|z|=1} \frac{(1+z)^n}{z^{n+1}} \\
&\qquad \times \frac{1}{2\pi i} \oint_{|w|=1} \frac{1}{w^{j+1}} \left(1 - (1+z)(1+w)\right)^n \, dw \, dz.
\end{aligned}$$

The inner integral recovers the jth coefficient in $\big(1-(1+z)(1+w)\big)^n = (-1)^n(z+(1+z)w)^n$, and we write

$$S_{n,j} = \frac{(-1)^n}{2\pi i}\oint_{|z|=1} \frac{(1+z)^n}{z^{n+1}}\binom{n}{j} z^{n-j}(1+z)^j\,dz$$
$$= \frac{(-1)^n}{2\pi i}\binom{n}{j}\oint_{|z|=1}\frac{(1+z)^{n+j}}{z^{j+1}}\,dz.$$

Then again, the remaining integral recovers the jth coefficient in $(1+z)^{n+j}$, yielding

$$S_{n,j} = (-1)^n\binom{n}{j}\binom{n+j}{j}.$$

Surprise, surprise, surprise!
All three methods give the same result!

A Moriarty identity: Egorychev II

Let us establish the identity

$$\sum_{k=m}^{n}(-4)^k\binom{k}{m}\binom{n+k}{2k}\frac{n}{n+k}=(-1)^n\,4^m\,\frac{n}{n+m}\binom{n+m}{2m}.$$

Toward a proof, let us call the sum $S_{n,m}$, and start with the simplification

$$\binom{n+k}{2k}\frac{n}{n+k}=\binom{n+k}{2k}\frac{n+k-k}{n+k}=\binom{n+k}{2k}-\frac{k}{n+k}\times\frac{n+k}{2k}\binom{n+k-1}{2k-1}.$$

We split $S_{n,m}$ into two sums: $S_{n,m}=S_{n,m}^{(1)}-S_{n,m}^{(2)}$, where

$$S_{n,m}^{(1)}:=\sum_{k=m}^{n}(-4)^k\binom{k}{m}\binom{n+k}{2k}=\sum_{k=m}^{n}(-4)^k\binom{k}{m}\binom{n+k}{n-k};$$

$$S_{n,m}^{(2)}:=\frac{1}{2}\sum_{k=m}^{n}(-4)^k\binom{k}{m}\binom{n+k-1}{2k-1}.$$

The first sum

via the index shift, we get

$$
\begin{aligned}
S^{(1)}_{n,m} &= \sum_{k=m}^{n} (-4)^k \binom{k}{m} \binom{n+k}{n-k} \\
&= \sum_{k=m}^{n} (-4)^k \binom{k}{m} [z^{n-k}] \, (1+z)^{n+k} \\
&= \sum_{k=m}^{n} (-4)^k \binom{k}{m} [z^{n}] \, z^k (1+z)^{n+k} \\
&= [z^n] \, (1+z)^n \sum_{k=m}^{n} (-4)^k \binom{k}{m} \, z^k (1+z)^k .
\end{aligned}
$$

In the summand, the least power of z is k. So, we can extend the sum to infinity and coefficient extractor provides 0 for the terms z^k, with $k > n$. In the following derivation, we re-index, setting $k = m + j$, then switch the sign of the upper index of the binomial coefficient:

$$
\begin{aligned}
S^{(1)}_{n,m} &= [z^n]\,(1+z)^n \sum_{j=0}^{\infty} (-4)^{m+j} \binom{m+j}{m} z^{m+j} (1+z)^{m+j} \\
&= (-4)^m\,[z^n]\,z^m (1+z)^{n+m} \sum_{j=0}^{\infty} (-4)^j \binom{m+j}{j} z^j (1+z)^j \\
&= (-4)^m\,[z^{n-m}]\,(1+z)^{n+m} \sum_{j=0}^{\infty} \binom{-m-1}{j} \big(4z(1+z)\big)^j .
\end{aligned}
$$

We have identified a part that can be summed by the binomial theorem in closed form:

$$
\begin{aligned}
S^{(1)}_{n,m} &= (-4)^m\,[z^{n-m}]\,(1+z)^{n+m} \frac{1}{(1+4z(1+z))^{m+1}} \\
&= (-4)^m \operatorname*{res}_z \frac{1}{z^{n-m+1}}\,(1+z)^{n+m} \frac{1}{(1+2z)^{2m+2}} .
\end{aligned}
$$

Egorychev's residue calculus

Put $z = g(w) = w/(1-w)$. Note that we are switching from z to w, while Rule (v) in the residue calculus presents a switch from w to z. We switch the roles of z and w in Rule (v), and use it at $k = n - m$, with the functions

$$f(z) = 1 + z, \qquad h(z) = \frac{z}{f(z)} = \frac{z}{1+z},$$

$$z = g(w) = \frac{w}{1-w}, \qquad A(z) = \frac{(1+z)^{2m}}{(1+2z)^{2m+2}}.$$

Note that the formal power series expansion of f has a nonzero free term (as $f(0) = 1$). So, in the numerator we have $A(g(w)) = A(w/(1-w)) = (1-w)^2/(1+w)^{(2m+2)}$, and in the denominator we have $f(g(w)) = 1 + w/(1-w) = 1/(1-w)$, and $h'(g(w)) = \frac{1}{(1+z)^2}\Big|_{z=w/(1-w)} = (1-w)^2$.

Executing Rule (v)

$$
\begin{aligned}
S^{(1)}_{n,m} &= \frac{A\big(g(w)\big)}{w^{n-m+1} f\big(g(w)\big)\, h'\big(g(w)\big)} \\
&= (-4)^m \operatorname*{res}_w \frac{1}{w^{n-m+1}} (1-w) \frac{(1-w)^2}{(1+w)^{2m+2}} \times \frac{1}{(1-w)^2} \\
&= (-4)^m \operatorname*{res}_w \frac{1}{w^{n-m+1}} \left(\frac{1-w}{(1+w)^{2m+2}} \right).
\end{aligned}
$$

The second sum is handled similarly:

$$
S^{(2)}_{n,m} = (-4)^m \operatorname*{res}_w \frac{1}{w^{n-m+1}} \left(\frac{\frac{1}{2}(1-w)^2}{(1+w)^{2m+2}} \right).
$$

Combing the parts

Using the switch of the sign in the upper index of the binomial coefficient, we get

$$
\begin{aligned}
S_{n,m} &= (-4)^m \operatorname*{res}_w \frac{1}{w^{n-m+1}} \left(\frac{(1-w) - \frac{1}{2}(1-w)^2}{(1+w)^{2m+2}} \right) \\
&= (-4)^m \operatorname*{res}_w \frac{1}{w^{n-m+1}} \left(\frac{\frac{1}{2}(1+w)(1-w)}{(1+w)^{2m+2}} \right) \\
&= \frac{1}{2} (-4)^m \operatorname*{res}_w \frac{1}{w^{n-m+1}} \left(\frac{1-w}{(1+w)^{2m+1}} \right) \\
&= \frac{1}{2} (-4)^m \left(\binom{-2m-1}{n-m} - \binom{-2m-1}{n-m-1} \right)
\end{aligned}
$$

Finally

$$\begin{aligned}
S_{n,m} &= \frac{1}{2}(-4)^m \left(\binom{-2m-1}{n-m} - \binom{-2m-1}{n-m-1} \right) \\
&= \frac{1}{2}(-4)^m \left((-1)^{n-m} \binom{n+m}{n-m} - (-1)^{n-m-1} \binom{n+m-1}{n-m-1} \right) \\
&= \frac{1}{2}(-1)^n 4^m \left(\binom{n+m}{n-m} + \frac{n-m}{n+m} \binom{n+m}{n-m} \right) \\
&= (-1)^n 4^m \binom{n+m}{2m} \frac{n}{n+m}.
\end{aligned}$$

Summing up

Egorychev method is

Easiest to use when the core function is a formal power series

$$a_0 + a_1 z + a_2 z^2 + \dots .$$

starting at power 0.
If the core has a Laurent series, such as for example

$$\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots ,$$

The residue calculus or Egorychev III may be the way out.

A Pólya-urn-like thank-you

Thank you.