A class of cycle type restricted permutation

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We propose to count a type of permutation where some integer k exists such that the set of cycle types in the permutation is $[k]$ i.e, the permutation contains cycles of size $1, 2, \ldots$ up to k. We will index these by k and call them type T in this document. We introduce a generating function using combinatorial classes as in *Analytic Combinatorics*

$$Q_k(z) = \prod_{q=1}^{k} \left[\exp\left(\frac{z^q}{q}\right) - 1\right].$$

where $Q_0(z) = 1.$ Here we have used the class

$$\textsc{Set}_{\ge 1}(\textsc{Cyc}_{=q}(\mathcal{Z})) \quad \text{which gives the EGF} \quad \exp\left(\frac{z^q}{q}\right) - 1.$$

This generating function counts permutations with cycle set where the first k types actually appear. It then follows that the mixed generating function of type T permutations is given by

$$Q(z, w) = \sum_{k\ge 0} w^k Q_k(z).$$

We set the upper range on k to n when employing this function. We will present a recurrence however in order to be able to effectively work with these quantities. A closely related statistic is permutations with k distinct cycle types. This is tabulated at OEIS A218868. The present sequence is a subset of these with the set of types instantiated to $[k]$. We can get the count of type T permutations by setting $w = 1$ and obtain the sequence

$$1, 1, 4, 7, 26, 181, 652, 3459, 22780, 265591, 1546436, 13294117, \ldots$$

The coefficient $F_n(w) = [z^n]Q(z, w)$ gives the EGF of these permutations on n letters by the number of distinct cycle types. For example with $n = 7$ we get

$$F_7(w) = \frac{1}{5040}w + \frac{11}{240}w^2 + \frac{1}{12}w^3.$$

As this is an EGF we scale by $n!$ and obtain e.g. for $n = 10$ the finite sequence

$$10!F_{10}(w) = w + 8550w^2 + 105840w^3 + 151200w^4.$$

Computing a recurrence

With the goal being to compute the coefficient $\rho_{n,k} = [z^n][w^k]Q(z, w)$ we can extract the coefficient on $[w^k]$ by inspection and are left with

$$\begin{aligned}\rho_{n,k} &= [z^n] \prod_{q=1}^{k} \left[\exp\left(\frac{z^q}{q}\right) - 1\right] \\ &= -\rho_{n,k-1} + [z^n] \exp\left(\frac{z^k}{k}\right) \prod_{q=1}^{k-1} \left[\exp\left(\frac{z^q}{q}\right) - 1\right] \\ &= -\rho_{n,k-1} + \sum_{p=0}^{\lfloor n/k \rfloor} [z^{kp}] \exp\left(\frac{z^k}{k}\right) [z^{n-kp}] \prod_{q=1}^{k-1} \left[\exp\left(\frac{z^q}{q}\right) - 1\right] \\ &= -\rho_{n,k-1} + \sum_{p=0}^{\lfloor n/k \rfloor} \left([z^p] \exp\left(\frac{z}{k}\right)\right) \rho_{n-kp,k-1} \\ &= -\rho_{n,k-1} + \sum_{p=0}^{\lfloor n/k \rfloor} \frac{1}{p! \times k^p} \rho_{n-kp,k-1} = \sum_{p=1}^{\lfloor n/k \rfloor} \frac{\rho_{n-kp,k-1}}{p! \times k^p}.\end{aligned}$$

The base case here is $\rho_{n,0} = \delta_{n,0}$. With this recurrence we can easily compute these values even for large n and the initial segment of the data yields the following triangular array:

1				
1				
1	3			
1	6			
1	25			
1	60	120		
1	231	420		
1	658	2800		
1	2619	20160		
1	8550	105840	151200	
1	35695	679140	831600	
1	129756	4848360	8316000	
1	568503	30356040	86486400	
1	2255344	217136920	1089728640	
1	10349535	1651409760	8967558600	10897286400

Here we note that for the number of distinct cycle types being k we must have $n \ge \frac{1}{2}k(k+1)$ or $1 \le k \le \lfloor (\sqrt{1+8n} - 1)/2 \rfloor$.

A conjecture

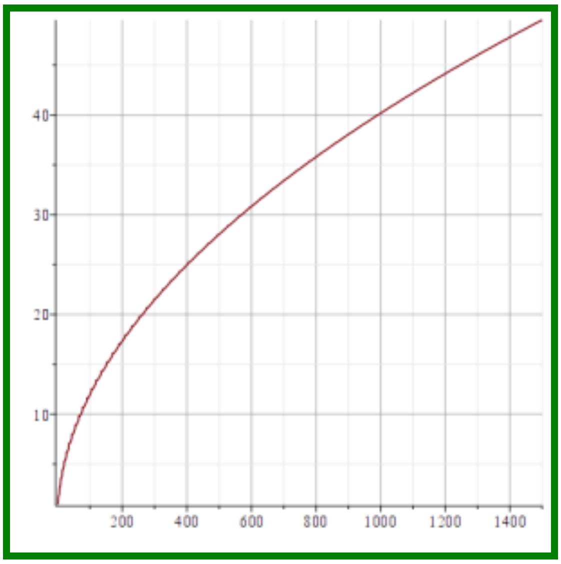
We introduce a random variable X giving the number of distinct cycle types in a type T permutation i.e. k and ask about the expectation. We may use the formula

$$E[X] = \frac{\frac{d}{dw}[z^n]Q(z,w)\Big|_{w=1}}{[z^n]Q(z,w)\big|_{w=1}}.$$

The numerics provide evidence that

$$\boxed{E[X] \sim N\sqrt{n}}$$

where $N \approx 1.27$. There might be a lower order term in n on the constant. The graph below illustrates this behavior where the data are displayed as what the recurrence produces.



Another interesting problem is the asymptotics of the count of type T permutations. A close match also obtained from numerics was $n! \times e^{-1.218 n^{2/3}}$. It is supposed that there may be lower order terms in the exponent.