A class of cycle type restricted permutation

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We propose to count a type of permutation where some integer k exists such that the set of cycle types in the permutation is $[k]$ i.e, the permutation contains cycles of size $1, 2, \ldots$ up to k. We will index these by the number of cycles and call them type T in this document. We introduce a generating function using combinatorial classes as in *Analytic Combinatorics*

$$Q_p(z, w) = \prod_{q=1}^{p} \left[\exp\left(w \frac{z^q}{q} \right) - 1 \right].$$

where $Q_0(z, w) = 1.$ Here we have used the class

$$\textsc{Set}_{\ge 1}(\textsc{Cyc}_{=q}(\mathcal{Z})) \quad \text{which gives the EGF} \quad \exp\left(\frac{z^q}{q} \right) - 1.$$

This generating function counts permutations with cycle set where the first k types actually appear, with the secondary variable giving the number of cycles. It then follows that the mixed generating function of type T permutations is given by

$$Q(z, w) = \sum_{p \ge 0} Q_p(z, w).$$

We set the upper range on p to n when employing this function. We will present a recurrence however in order to be able to effectively work with these quantities. We can get the count of type T permutations by setting $w = 1$ and obtain the sequence

$$1, 1, 4, 7, 26, 181, 652, 3459, 22780, 265591, 1546436, 13294117, \ldots$$

The coefficient $F_n(w) = [z^n] Q(z, w)$ gives the EGF by the number of cycles of type T permutations, for example we have

$$F_{11}(w) = \frac{1}{39916800} w^{11} + \frac{1}{725760} w^{10} + \frac{1}{40320} w^9 + \frac{7}{17280} w^8 + \frac{5}{2304} w^7 + \frac{289}{34560} w^6 + \frac{1}{36} w^5.$$

This being an EGF we get e.g. the finite sequence

$$12! F_{12}(w) = w^{12} + 66 w^{11} + 1485 w^{10} + 29700 w^9 + 218295 w^8 + 1171170 w^7 + 5405400 w^6 + 6468000 w^5.$$

Computing a recurrence

With the goal being to compute the coefficient $\rho_{n,k} = [z^n][w^k] Q(z, w)$ we first introduce $\nu_{n,k,p} = [z^n][w^k] Q_p(z, w)$ and extract one term from the product:

$$\begin{aligned}
\nu_{n,k,p} &= [z^n][w^k] \prod_{q=1}^{p} \left[\exp\left(w\frac{z^q}{q}\right) - 1\right] \\
&= -\nu_{n,k,p-1} + [z^n][w^k] \exp\left(w\frac{z^p}{p}\right) \prod_{q=1}^{p-1} \left[\exp\left(w\frac{z^q}{q}\right) - 1\right] \\
&= -\nu_{n,k,p-1} + [w^k] \sum_{r=0}^{\lfloor n/p \rfloor} \left([z^{pr}] \exp\left(w\frac{z^p}{p}\right)\right) [z^{n-pr}] Q_{p-1}(z,w) \\
&= -\nu_{n,k,p-1} + [w^k] \sum_{r=0}^{\lfloor n/p \rfloor} \left([z^{r}] \exp\left(w\frac{z}{p}\right)\right) [z^{n-pr}] Q_{p-1}(z,w) \\
&= -\nu_{n,k,p-1} + [w^k] \sum_{r=0}^{\lfloor n/p \rfloor} \frac{w^r}{r! \times p^r} [z^{n-pr}] Q_{p-1}(z,w) \\
&= -\nu_{n,k,p-1} + \sum_{r=0}^{\min(k,\lfloor n/p \rfloor)} \frac{1}{r! \times p^r} [w^{k-r}][z^{n-pr}] Q_{p-1}(z,w) \\
&= -\nu_{n,k,p-1} + \sum_{r=0}^{\min(k,\lfloor n/p \rfloor)} \frac{1}{r! \times p^r} \nu_{n-pr,k-r,p-1} = \sum_{r=1}^{\min(k,\lfloor n/p \rfloor)} \frac{\nu_{n-pr,k-r,p-1}}{r! \times p^r}.
\end{aligned}$$

The base case here is $\nu_{n,k,0} = \delta_{n,0}\delta_{k,0}$. We also have $\nu_{n,k,p} = 0$ when $n < 1/2 \times p(p+1)$ or $k < p$. We then obtain with $M = \min(\lfloor(\sqrt{1+8n} - 1)/2\rfloor, k)$

$$\rho_{n,k} = \sum_{p=0}^{n} \nu_{n,k,p} = \sum_{p=0}^{M} \nu_{n,k,p}.$$

With this recurrence we can easily compute these values even for large n and the initial segment of the data yields the following triangular array:

1								
1								
3	1							
6	1							
15	10	1						
120	45	15	1					
525	105	21	1					
1680	1540	210	28	1				
10080	8505	3780	378	36	1			
151200	75600	29925	8190	630	45	1		
1108800	333795	86625	16170	990	55	1		
6468000	5405400	1171170	218295	29700	1485	66	1	
43243200	49249200	20855835	3513510	495495	51480	2145	78	1

The minimum number of cycles counted by $F_n(w)$ occurs if we fit the maximal k into n and fill the rest with one additional cycle if necessary. This gives $n \ge k \ge \lceil (\sqrt{1+8n}-1)/2 \rceil$ for the range of k as applied in the above table.

A conjecture

We introduce a random variable X giving the number of cycles in a type T permutation i.e. k and ask about the expectation. We may use the formula

$$\mathrm{E}[X] = \frac{\frac{d}{dw}[z^n]Q(z,w)\Big|_{w=1}}{[z^n]Q(z,w)|_{w=1}}.$$

The numerics provide evidence that

$$\boxed{\mathrm{E}[X] \sim N\sqrt{n}}$$

where $N \approx 1.57.$ There might be a lower order term in n on the constant. The graph below illustrates this behavior where the data are displayed as what the recurrence produces.