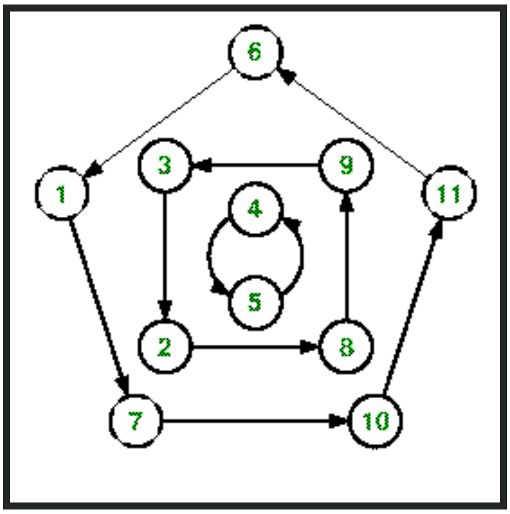
Nested Cycle Partitions: a conjecture

Marko Riedel

January 2026

In this document we propose to investigate a combinatorial class $\mathrm{NCYC}_{=k}$ in the spirit of *Analytic Combinatorics* by P. Flajolet, where the NC stands for NESTED CYCLE. Everything that follows takes place in the labeled universe of exponential generating functions. An element of $\mathrm{NCYC}_{=k}$ consists of an outer cycle of length k, with additional cycles nested inside with the constraint that they fit into one another, i.e. the numbers of nodes of the constituent cycles form a strictly decreasing sequence of natural numbers starting with k. Alternatively we may observe that with two adjacent constituent cycles the outer one has more nodes. An example is shown below:



This shows an element of $\mathrm{NCYC}_{=5}$ which corresponds to the sequence $5, 4, 2$.

Nested Cycle Partitions

Analogous to Stirling Cycle Numbers (Stirling Numbers of the First Kind) which partition n nodes into a set of ordinary cycles, we define a Nested Cycle Partition on n nodes to be a set of nested cycles where the total number of all nodes is n. This document includes two examples which follow below. These NCPs thus have combinatorial class specification

$$\textsc{Set}(\mathcal{U} \times \mathrm{NCYC}_{=1}(\mathcal{Z}) + \mathcal{U} \times \mathrm{NCYC}_{=2}(\mathcal{Z}) + \mathcal{U} \times \mathrm{NCYC}_{=3}(\mathcal{Z}) + \cdots).$$

Here we use $\mathcal{U}$ to mark the number of components and $\mathcal{Z}$ the number of nodes. Introducing the generating function $Q_k(z)$ of $\mathrm{NCYC}_{=k}(\mathcal{Z})$ we thus have the EGF $Q(z,u)$ of NCPs by the number of nodes and components

$$Q(z,u) = \exp\left(u \sum_{k\geq 1} Q_k(z)\right).$$

What can we say about $Q_k(z)$? We have from first principles that with $Q_1(z) = z$ the following recurrence holds:

$$Q_k(z) = \frac{z^k}{k}\left(1 + \sum_{p=1}^{k-1} Q_p(z)\right).$$

Also from first principles we find that

$$Q_k(z) = \frac{z^k}{k} \prod_{p=1}^{k-1} \left(1 + \frac{z^p}{p}\right).$$

Setting $u = 1$ to obtain the count from $Q(z, 1)$ and using that (all three properties follow immediately by inspection)

$$\sum_{k\ge 1} Q_k(z) = -1 + \prod_{p\ge 1} \left(1 + \frac{z^p}{p}\right).$$

we obtain the sequence

$$1, 2, 9, 44, 270, 2064, 17682, 171296, 1867968, 22470840, 294493320, \\ 4195969392, 64416698112, 1059685905264, 18609306423120, \ldots$$

This points to OEIS A308338 where at the time of writing no additional data was found.

The coefficients of $\sum_{k\ge 1} Q_k(z)$

Seeking

$$\kappa_n = [z^n] \sum_{k\ge 1} Q_k(z)$$

we differentiate to get

$$\kappa_n = \frac{1}{n} [z^{n-1}] \prod_{p\ge 1} \left(1 + \frac{z^p}{p}\right) \sum_{p\ge 1} \frac{z^{p-1}}{1 + z^p/p} \\ = \frac{1}{n} \sum_{m=0}^{n-1} [z^{n-1-m}] \prod_{p\ge 1} \left(1 + \frac{z^p}{p}\right) [z^m] \sum_{p=1}^{m+1} \frac{z^{p-1}}{1 + z^p/p}.$$

Now we have for the extractor in $[z^m]$

$$\sum_{p=1}^{m+1} [z^{m+1-p}] \frac{1}{1 + z^p/p} = \sum_{p|m+1} [z^{(m+1)/p-1}] \frac{1}{1 + z/p} = \sum_{p|m+1} (-1/p)^{(m+1)/p-1}.$$

Call this quantity ν_{m+1}. Returning to the recurrence

$$\kappa_n = \frac{1}{n} \sum_{m=0}^{n-2} \kappa_{n-1-m} \nu_{m+1} + \frac{1}{n} \nu(n).$$

We also seek

$$\rho_n = [z^n] \exp\left(\sum_{k\ge 1} Q_k(z)\right)$$

Differentiating

$$\rho_n = \frac{1}{n}[z^{n-1}] \exp\left(\sum_{k\ge 1} Q_k(z)\right) \sum_{k\ge 1} \frac{d}{dz} Q_k(z)$$
$$= \frac{1}{n}\sum_{m=0}^{n-1} [z^{n-1-m}] \exp\left(\sum_{k\ge 1} Q_k(z)\right) [z^m] \frac{d}{dz} \sum_{k\ge 1} Q_k(z)$$
$$= \frac{1}{n}\sum_{m=0}^{n-1} \rho_{n-1-m} (m+1)[z^{m+1}] \sum_{k\ge 1} Q_k(z)$$
$$= \frac{1}{n}\sum_{m=0}^{n-1} \rho_{n-1-m} (m+1)\kappa_{m+1}.$$

The conjecture

We ask about the expected number of components in a random NCP on n nodes which we denote by the random variable X. We this in mind we differentiate $Q(z, u)$ with respect to u and set $u = 1$ to obtain for the sum of this statistic over all NCPs

$$n![z^n] \exp\left(\sum_{k\ge 1} Q_k(z)\right) \sum_{k\ge 1} Q_k(z).$$

We can now use the recurrences for κ_n and ρ_n to compute (factorials cancel)

$$\mathrm{E}[X] = \frac{[z^n] \frac{d}{du} Q(z,u)\Big|_{u=1}}{[z^n] Q(z,1)}$$

where X is the number of components including for large n using memoization on those recurrences. Using these data the conjecture becomes

$$\boxed{\mathrm{E}[X] \sim N\sqrt{n}.}$$

where N is a constant that is to be determined. If you think you can contribute to an answer, then contact the author. We might have asymptotics for a term n^M with M some constant close to but not equal to $1/2.$ The numeric data suggest that $M \ge 0.4704567259$ and that $N \approx 0.7719351399.$

Counting NCPs with some number j of components

Here we seek

$$\rho_{n,j} = [z^n] \frac{1}{j!}\left(\sum_{k\ge 1} Q_k(z)\right)^j.$$

where $\rho_{n,0} = \delta_{n,0}$. Differentiating we find

$$\rho_{n,j} = \frac{1}{n}[z^{n-1}] \frac{1}{(j-1)!}\left(\sum_{k\ge 1} Q_k(z)\right)^{j-1} \sum_{k\ge 1} \frac{d}{dz} Q_k(z)$$
$$= \frac{1}{n}\sum_{m=0}^{n-1} [z^{n-1-m}] \frac{1}{(j-1)!}\left(\sum_{k\ge 1} Q_k(z)\right)^{j-1} [z^m] \sum_{k\ge 1} \frac{d}{dz} Q_k(z)$$
$$= \frac{1}{n}\sum_{m=0}^{n-1} \rho_{n-1-m,j-1} (m+1)\kappa_{m+1}.$$

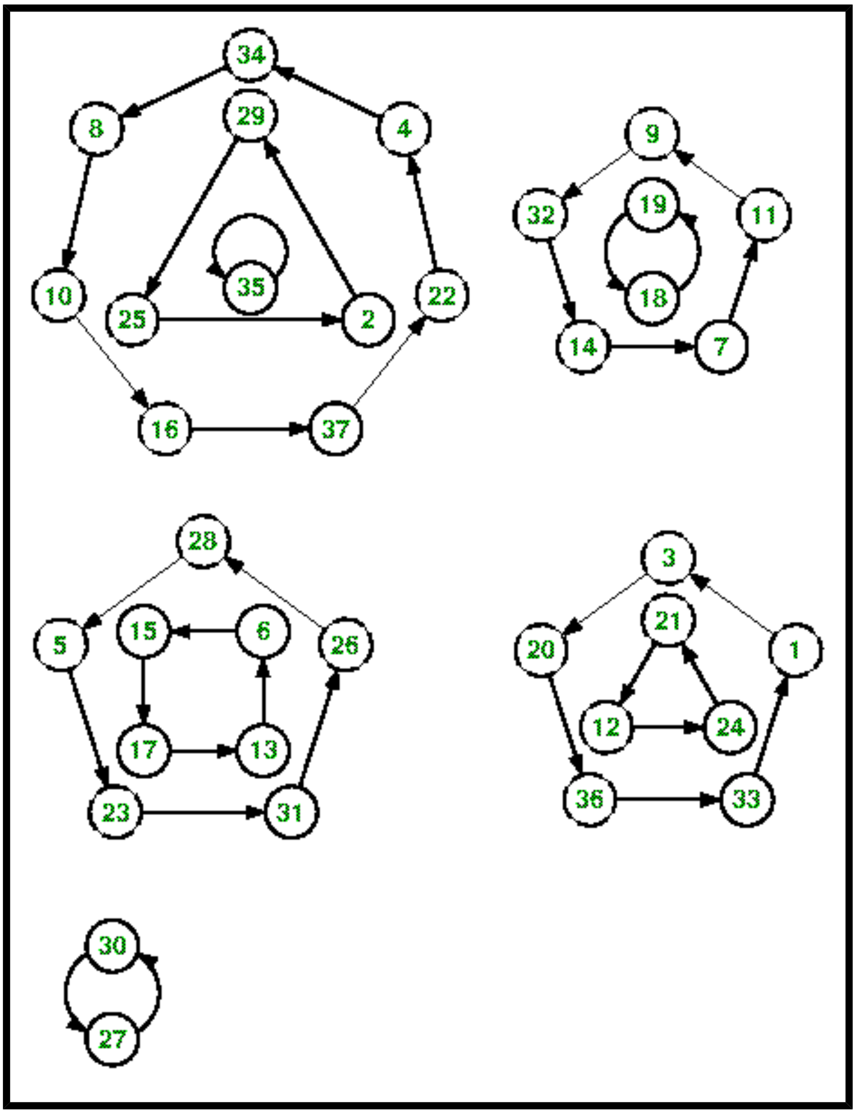
This gives the following triangular array (exponential generating function $n!\rho_{n,j}$)

1									
1	1								
5	3	1							
14	23	6	1						
74	120	65	10	1					
474	904	525	145	15	1				
3114	7322	5299	1645	280	21	1			
24240	65764	55244	21329	4200	490	28	1		
219456	659880	630944	279720	67809	9324	798	36	1	
2231280	7237296	7782720	3925340	1091265	182973	18690	1230	45	1

Observe that the leftmost entries point to OEIS A007838, permutations with distinct cycle lengths. This is indeed the right semantics because when we have just one component it precisely provides a set of cycles of distinct lengths. BTW we may verify that these give the row sums of all partitions on n nodes that we saw earlier.

Example A

This example shows an NCP on 37 nodes and five components.



Example B

This example shows an NCP on 40 nodes and four components.