Measuring post–quickselect disorder

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Abstract

This paper deals with the amount of disorder that is left in a permutation after one of its elements has been selected with quickselect or quickselect with median-of-three pivoting. Five measures of disorder are considered: inversions, cycles of length less than or equal to some m, cycles of any length, expected cycle length, and the distance to the identity permutation. "Grand averages" for each measure of disorder for a permutation after one of its elements has been selected with quickselect, where $1, 2, \ldots, n$ are the elements being permuted, are computed, as well as more specific results.

KEYWORDS. Quickselect, inversions, cycles, disorder.

1 Introduction

Quickselect (sometimes called Hoare's FIND algorithm) is an algorithm that has been extensively studied and uses the principle behind the quicksort algorithm to select one or more elements from a permutation [2, 3, 5, 9]. The goal is to select an order statistic from a permutation. The algorithm selects a pivot (this is either the first element or the median of the first three elements in quickselect with median-of-three pivoting) and splits the data into those elements that are less than the pivot, those that are equal to the pivot and those that are larger than the pivot. If the pivot is the statistic that we wish to find, the algorithm halts. Otherwise it recursively selects the desired order statistic from those elements that are less than, or those that are larger than the pivot.

This paper treats the following question. What amount of disorder is left in a permutation after one of its elements has been selected with quickselect or quickselect with median-of-three pivoting? It seems clear that there should be less disorder than in a random permutation. Those elements that were pivots are in place, and the others are closer to their home position than before quickselect was applied to the permutation. We consider five measures of disorder:

- **Inversions.** Two elements of a permutation such that the first is larger than the second constitute an inversion. The fewer inversions, the more ordered the permutation.
- **Cycles of length less than or equal to some m.** The value $m = 1$ is of particular interest, because it counts the number of fixed points. The more cycles, the more ordered the permutation.
- **Cycles of any length.** This is like the previous item, except that now all cycles are counted. Once again, the more cycles, the more ordered the permutation.
- **Expected cycle length.** Pick a random element of a random permutation. It belongs to a cycle of some length k. We study the expected value of k. This parameter should decrease after processing by quickselect.
- **Distance to the identity permutation.** Sum the absolute value of the distance of each element to its correct position, taken to some power p. We treat the case $p = 2$. The smaller the distance, the more ordered the permutation.

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The goal of this paper is to compute the "grand average" for each measure of disorder for a permutation after one of its elements has been selected with quickselect, where $1, 2, \ldots, n$ are the elements being permuted. More specific results are obtained in the process of computing these "grand averages."

We have used Maple, Maxima, GiNaC and Perl to compute the results of this paper.

2 Random permutations

We must first compute the expected value of each measure for random permutations of n elements, because these expectations enter the computations, as will become clear later. We can compare the value of this quantity with the corresponding one after quickselect has been performed; this will be summarized in Section 7. For the computations of the variance as done in Section 6 we also need the second factorial moments for these measures.

Let $r_n(v)$ be the exponential generating function of any such measure and set

$$R(z, v) = \sum_{n\geq 1} r_n(v) z^n.$$

We always have $r_n(1) = 1$ and

$$R(z, 1) = \frac{1}{1-z}.$$

2.1 Distance to the identity permutation

This is a less common measure and requires commentary. For $p \geq 1$ the distance $d_p(\pi)$ is for a permutation $\pi_1\pi_2 \ldots \pi_n$ of size n given by

$$d_p(\pi) := \sum_{k=1}^{n} |k - \pi_k|^p. \tag{1}$$

In this case $r_{n,p}(v)$ is the exponential generating function of the random variable

$$R_n := [\text{distance to the identity permutation of a random permutation of size } n].$$

We have

$$\mathbb{E}(R_n) = \frac{1}{n} \sum_{1\leq k,a\leq n} |k - a|^p \tag{2a}$$

and

$$\mathbb{E}(R_n^2) = \frac{1}{n} \sum_{1\leq k,a\leq n} |k - a|^{2p} + \frac{1}{n(n-1)} \sum_{\substack{1\leq k,l,a,b\leq n \\ k\neq l, a\neq b}} |k - a|^p |l - b|^p. \tag{2b}$$

These formulæ can be obtained by averaging (1):

$$\mathbb{E}(R_n) = \frac{1}{n!} \sum_{\pi\in S_n} d_p(\pi) = \frac{1}{n!} \sum_{\pi\in S_n} \sum_{k=1}^{n} |k - \pi_k|^p = \frac{1}{n!} \sum_{k=1}^{n} \sum_{a=1}^{n} \sum_{\pi\in S_n, \pi_k=a} |k - a|^p = \frac{1}{n} \sum_{1\leq k,a\leq n} |k - a|^p$$

and

$$\begin{aligned}
\mathbb{E}(R_n^2) &= \frac{1}{n!} \sum_{\pi\in S_n} d_p^2(\pi) = \frac{1}{n!} \sum_{\pi\in S_n} \left(\sum_{k=1}^{n} |k - \pi_k|^{2p} + \sum_{\substack{1\leq k,l\leq n \\ k\neq l}} |k - \pi_k|^p |l - \pi_l|^p \right) \\
&= \frac{1}{n} \sum_{1\leq k,a\leq n} |k - a|^{2p} + \frac{1}{n!} \sum_{\substack{1\leq k,l\leq n \\ k\neq l}} \sum_{\substack{\pi\in S_n \\ \pi_k=a, \pi_l=b}} |k - \pi_k|^p |l - \pi_l|^p \\
&= \frac{1}{n} \sum_{1\leq k,a\leq n} |k - a|^{2p} + \frac{1}{n(n-1)} \sum_{\substack{1\leq k,l,a,b\leq n \\ k\neq l, a\neq b}} |k - a|^p |l - b|^p.
\end{aligned}$$

¿From (2a) we get then

$$r'_{n,p}(1) = \frac{2}{n} \sum_{k=1}^{n-1} k^p \, (n-k).$$

Recall that (e. g. [1])

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n, \qquad \sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2,$$
$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n, \qquad \sum_{k=1}^{n} k^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.$$

This yields to

$$r'_{n,1}(1) = \frac{(n+1)(n-1)}{3}, \qquad r'_{n,2}(1) = \frac{(n+1)n(n-1)}{6},$$
$$r'_{n,3}(1) = \frac{(n+1)(n-1)(3n^2-2)}{30}, \qquad r'_{n,4}(1) = \frac{(n+1)n(n-1)(2n^2-3)}{30}, \qquad \text{etc.}$$

Such computations are best done with computer algebra systems.

2.2 Considered measures for random permutations

We now list $r_n(v)$ for the five measures under consideration.

- **Inversions.**

$$r_n(v) = \frac{1}{n!} \prod_{k=0}^{n-1} \sum_{l=0}^{k} v^l = \frac{1}{n!} \prod_{k=0}^{n-1} \frac{1-v^{k+1}}{1-v},$$
$$R(z,v) = \sum_{n \geq 0} r_n(v) z^n = \sum_{n \geq 0} \frac{z^n (v;v)_n}{n!(1-v)^n},$$

where we used the notation $(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1})$. Note that

$$r'_n(1) = \frac{n(n-1)}{4}, \quad r''_n(1) = \frac{n(n-1)(n-2)(9n+13)}{144}$$

and hence

$$\left.\frac{\partial}{\partial v} R(z,v)\right|_{v=1} = \frac{1}{4} z^2 \sum_{n \geq 2} n(n-1) z^{n-2} = \frac{1}{4} z^2 \left(\frac{1}{1-z} \right)'' = \frac{1}{2} \frac{z^2}{(1-z)^3},$$
$$\left.\frac{\partial^2}{\partial v^2} R(z,v)\right|_{v=1} = \frac{z^3(10-z)}{6(1-z)^5}.$$

- **Cycles of length less than or equal to some m.** For these generating functions we refer to [9]:

$$R(z,v) = \exp\left(v \sum_{k=1}^{m} \frac{z^k}{k} + \sum_{k=m+1}^{\infty} \frac{z^k}{k} \right) = \frac{1}{1-z} \exp\left((v-1) \sum_{k=1}^{m} \frac{z^k}{k} \right)$$

and thus

$$\left.\frac{\partial}{\partial v} R(z,v)\right|_{v=1} = \frac{z + z^2/2 + z^3/3 + \ldots + z^m/m}{1-z}.$$

Note the special instance $m = 1$ (counting fixed points):

$$r_n(v) = [z^n]\frac{e^{vz-z}}{1-z}$$

which gives

$$\left.\frac{\partial}{\partial v}R(z,v)\right|_{v=1} = \frac{z}{1-z} \quad \text{and} \quad \left.\frac{\partial^2}{\partial v^2}R(z,v)\right|_{v=1} = \frac{z^2}{1-z}.$$

- **Cycles of any length.** Here we have

$$R(z,v) = \frac{1}{(1-z)^v}$$

and thus

$$r_n(v) = [z^n]\left(\frac{1}{1-z}\right)^v = [z^n]\, e^{v \log \frac{1}{1-z}}$$

and

$$\left.\frac{\partial}{\partial v}R(z,v)\right|_{v=1} = \frac{1}{1-z}\log\frac{1}{1-z}, \quad \left.\frac{\partial^2}{\partial v^2}R(z,v)\right|_{v=1} = \frac{1}{1-z}\left(\log\frac{1}{1-z}\right)^2.$$

- **Expected cycle length.** We have

$$R(z,v) = \exp\left(\sum_{k\ge 1} v^{k^2}\frac{z^k}{k}\right)$$

and hence

$$\left.\frac{\partial}{\partial v}R(z,v)\right|_{v=1} = \frac{1}{1-z}\left(\sum_{k\ge 1} k^2\frac{z^k}{k}\right) = \frac{z}{(1-z)^3},$$
$$\left.\frac{\partial^2}{\partial v^2}R(z,v)\right|_{v=1} = \frac{7z^2}{(1-z)^5}.$$

- **Distance to the identity permutation.** We confine ourselves to the parameter $p = 2$.

$$\left.\frac{\partial}{\partial v}R(z,v)\right|_{v=1} = \sum_{n\ge 0}\binom{n+1}{3}z^n = \frac{z^2}{(1-z)^4}.$$

¿From (2b) one also gets

$$\left.\frac{\partial^2}{\partial v^2}R(z,v)\right|_{v=1} = \frac{z^2(5z^3+13z^2+39z+3)}{3(1-z)^7}.$$

3 Recurrence relation

Let $q_{n,j}(v)$ be the exponential generating function of the measure of disorder (measured by a certain random variable $Q_{n,j}$) after the element j, where $1 \le j \le n$, has been selected from a random permutation of size n.

The base cases are:

- **Inversions and distance to the identity permutation.** $q_{1,1}(v) = 1/1 = 1$ and $q_{2,1}(v) = q_{2,2}(v) = 2/2 = 1$.
- **Fixed points, expected cycle length, and cycles of any length.** $q_{1,1}(v) = v/1 = v$ and $q_{2,1}(v) = q_{2,2}(v) = 2v^2/2 = v^2$.

For convenience we let $r_0(v) = q_{0,j}(v) = 1$. Let p denote the pivot, where $1 \le p \le n$. The recurrence for $q_{n,j}(v)$ is as follows:

$$q_{n,j}(v) = \frac{1}{n}\sum_{p=1}^{j-1} r_{p-1}(v)q_{n-p,j-p}(v) + \frac{1}{n} r_{j-1}(v)r_{n-j}(v) + \frac{1}{n}\sum_{p=j+1}^{n} q_{p-1,j}(v)r_{n-p}(v) \tag{3a}$$

for inversions and the distance to the identity permutation and

$$q_{n,j}(v) = \frac{v}{n}\sum_{p=1}^{j-1} r_{p-1}(v)q_{n-p,j-p}(v) + \frac{v}{n} r_{j-1}(v)r_{n-j}(v) + \frac{v}{n}\sum_{p=j+1}^{n} q_{p-1,j}(v)r_{n-p}(v) \tag{3b}$$

for fixed points, the expected cycle length and cycles.

The recurrence for median-of-three pivoting is as follows (note that the probability of p being selected as the pivot is $\binom{n}{3}^{-1}(p-1)(n-p)$)

$$\begin{aligned} q_{n,j}(v) &= \binom{n}{3}^{-1} \sum_{p=1}^{j-1}(p-1)r_{p-1}(v)(n-p)q_{n-p,j-p}(v) \\ &+ \binom{n}{3}^{-1} (j-1)r_{j-1}(v)(n-j)r_{n-j}(v) \\ &+ \binom{n}{3}^{-1} \sum_{p=j+1}^{n}(p-1)q_{p-1,j}(v)(n-p)r_{n-p}(v) \end{aligned} \tag{3c}$$

for inversions and the distance to the identity permutation and

$$\begin{aligned} q_{n,j}(v) &= v\binom{n}{3}^{-1} \sum_{p=1}^{j-1}(p-1)r_{p-1}(v)(n-p)q_{n-p,j-p}(v) \\ &+ v\binom{n}{3}^{-1} (j-1)r_{j-1}(v)(n-j)r_{n-j}(v) \\ &+ v\binom{n}{3}^{-1} \sum_{p=j+1}^{n}(p-1)q_{p-1,j}(v)(n-p)r_{n-p}(v) \end{aligned} \tag{3d}$$

for fixed points, the expected cycle length and cycles.

For all these measures, except the expected cycle length, we will compute the expectations

$$E_{n,j} := \mathbb{E}(Q_{n,j}) = q'_{n,j}(1),$$

but in particular we are interested in

$$E_n := \frac{1}{n}\sum_{j=1}^{n} E_{n,j} = \frac{q'_n(1)}{n} \quad \text{where} \quad q_n(v) = \sum_{j=1}^{n} q_{n,j}(v),$$

which is the grand average of the parameter under consideration. In the case of the expected cycle length we have

$$E_{n,j} = \frac{1}{n}q'_{n,j}(1) \quad \text{and} \quad E_n = \frac{q'_n(1)}{n^2}.$$

For the "ordinary" Quickselect algorithm with splitting probabilities $\frac{1}{n}$ we also compute the second factorial moments, which is for all parameters except the expected cycle length given by

$$M^{(2)}_{n,j} := \mathbb{E}\left(Q_{n,j}(Q_{n,j}-1)\right) = q''_{n,j}(1),$$

and from which the variance

$$V_{n,j} := M^{(2)}_{n,j} + E_{n,j} - E^2_{n,j}$$

could be obtained easily. Interesting is also the variance of the mean

$$V_n := \frac{1}{n}\sum_{j=1}^{n} M^{(2)}_{n,j} + E_n - E_n^2,$$

which is given, too. For the expected cycle length things are a bit different; we have

$$M^{(2)}_{n,j} = \frac{1}{n^2} q''_{n,j}(1) - \frac{n-1}{n^2} q'_{n,j}(1)$$

and

$$V_n = \frac{1}{n^3}\left(q''_n(1) + q'_n(1)\right) - \left(\frac{1}{n^2} q'_n(1)\right)^2.$$

4 Trivariate generating function

To get our results for the considered parameters, we will use a generating functions approach. We will introduce trivariate generating functions

$$F(z,u,v) = \sum_{1\le j\le n} q_{n,j}(v) u^j z^n,$$

from which the recurrence relations (3) will translate into ordinary differential equations.

Recall that $q_{n,j}(1) = 1$ and hence

$$F(z,u,1) = \sum_{1\le j\le n} u^j z^n = \frac{1}{1-z}\sum_{1\le j}(zu)^j = \frac{zu}{(1-z)(1-zu)}.$$

¿From (3a) we get

$$\begin{aligned} n q_{n,j}(v) u^j z^{n-1} &= u \sum_{p=1}^{j-1} r_{p-1}(v) u^{p-1} z^{p-1} q_{n-p,j-p}(v) u^{j-p} z^{n-p} \\ &+ u r_{j-1}(v) u^{j-1} z^{j-1} r_{n-j}(v) z^{n-j} \\ &+ \sum_{p=j+1}^{n} q_{p-1,j}(v) u^j z^{p-1} r_{n-p}(v) z^{n-p} \end{aligned}$$

for inversions and the distance to the identity permutation and hence

$$\frac{\partial}{\partial z} F(z,u,v) = uR(zu,v)F(z,u,v) + uR(zu,v)R(z,v) + R(z,v)F(z,u,v).$$

In an analogous way we obtain from (3b) the differential equation

$$\frac{\partial}{\partial z} F(z,u,v) = uv\, R(zu,v)F(z,u,v) + uv\, R(zu,v)R(z,v) + v\, R(z,v)F(z,u,v)$$

for fixed points, the expected cycle length and cycles. Differentiate with respect to v to obtain

$$\begin{aligned} \frac{\partial}{\partial v}\frac{\partial}{\partial z} F(z,u,v) &= u\frac{\partial}{\partial v} R(zu,v)F(z,u,v) + uR(zu,v)\frac{\partial}{\partial v} F(z,u,v) \\ &+ u\frac{\partial}{\partial v} R(zu,v)R(z,v) + uR(zu,v)\frac{\partial}{\partial v} R(z,v) \\ &+ \frac{\partial}{\partial v} R(z,v)F(z,u,v) + R(z,v)\frac{\partial}{\partial v} F(z,u,v) \end{aligned}$$

for inversions and the distance to the identity permutation and

$$\frac{\partial}{\partial v}\frac{\partial}{\partial z} F(z,u,v) = uR(zu,v)F(z,u,v) + uv\frac{\partial}{\partial v} R(zu,v)F(z,u,v) + uvR(zu,v)\frac{\partial}{\partial v} F(z,u,v)$$

$$+ uR(zu,v)R(z,v) + uv\frac{\partial}{\partial v}R(zu,v)R(z,v) + uvR(zu,v)\frac{\partial}{\partial v}R(z,v)$$
$$+ R(z,v)F(z,u,v) + v\frac{\partial}{\partial v}R(z,v)F(z,u,v) + vR(z,v)\frac{\partial}{\partial v}F(z,u,v)$$

for fixed points, the expected cycle length and cycles. Let

$$G(z,u) = \left(\frac{\partial}{\partial v}F(z,u,v)\right)\bigg|_{v=1}, \quad \text{thus} \quad E_{n,j} = [z^n u^j]G(z,u)$$

for all considered parameters except the expected cycle length. Here we get

$$E_{n,j} = \frac{1}{n}[z^n u^j]G(z,u).$$

Differentiating with respect to v and evaluating at $v = 1$ leads in the instance of inversions to

$$\frac{\partial}{\partial z}G(z,u) = \frac{z^3u^4}{2(1-zu)^4(1-z)} + \frac{u}{1-zu}G(z,u) + \frac{z^2u^3}{2(1-zu)^3(1-z)}$$
$$+ \frac{z^2u}{2(1-zu)(1-z)^3} + \frac{z^3u}{(1-zu)(1-z)^4} + \frac{1}{1-z}G(z,u)$$

or

$$\frac{\partial}{\partial z}G(z,u) = \frac{1}{2}uz^2\frac{1-3uz-3u^2z+6u^2z^2-u^2z^3-u^3z^3+u^2}{(1-zu)^4(1-z)^4} + \frac{1+u-2zu}{(1-zu)(1-z)}G(z,u).$$

In the instance of fixed points we obtain

$$\frac{\partial}{\partial z}G(z,u) = \frac{zu^2}{(1-zu)^2(1-z)} + \frac{z^2u^3}{(1-zu)^2(1-z)} + \frac{u}{1-zu}G(z,u)$$
$$+ \frac{u}{(1-zu)(1-z)} + \frac{zu^2}{(1-zu)(1-z)} + \frac{zu}{(1-zu)(1-z)}$$
$$+ \frac{zu}{(1-zu)(1-z)^2} + \frac{z^2u}{(1-zu)(1-z)^2} + \frac{1}{1-z}G(z,u)$$

or

$$\frac{\partial}{\partial z}G(z,u) = \frac{u(1+z+uz-3uz^2)}{(1-zu)^2(1-z)^2} + \frac{1+u-2zu}{(1-zu)(1-z)}G(z,u).$$

In the instance of cycles we obtain

$$\frac{\partial}{\partial z}G(z,u) = \frac{zu^2}{(1-zu)^2(1-z)} + \frac{zu^2}{(1-zu)^2(1-z)}\log\frac{1}{1-zu} + \frac{u}{1-zu}G(z,u)$$
$$+ \frac{u}{(1-zu)(1-z)} + \frac{u}{(1-zu)(1-z)}\log\frac{1}{1-zu} + \frac{u}{(1-zu)(1-z)}\log\frac{1}{1-z}$$
$$+ \frac{zu}{(1-zu)(1-z)^2} + \frac{zu}{(1-zu)(1-z)^2}\log\frac{1}{1-z} + \frac{1}{1-z}G(z,u)$$

or

$$\frac{\partial}{\partial z}G(z,u) = \frac{u}{(1-zu)(1-z)^2}\log\frac{1}{1-z} + \frac{u}{(1-zu)^2(1-z)}\log\frac{1}{1-zu}$$
$$+ \frac{u-u^2z^2}{(1-zu)^2(1-z)^2} + \frac{1+u-2zu}{(1-zu)(1-z)}G(z,u).$$

For the expected cycle length, we obtain

$$\frac{\partial}{\partial z}G(z,u) = \frac{zu^2}{(1-z)(1-zu)^2} + \frac{z^2u^3}{(1-z)(1-zu)^4} + \frac{u}{1-zu}G(z,u)$$

$$+\frac{u}{(1-z)(1-zu)}+\frac{zu^2}{(1-z)(1-zu)^3}+\frac{zu}{(1-z)^3(1-zu)}$$
$$+\frac{zu}{(1-z)^2(1-zu)}+\frac{z^2u}{(1-z)^4(1-zu)}+\frac{1}{1-z}G(z,u)$$

or

$$\frac{\partial}{\partial z}G(z,u)=-u\frac{u^3z^6-2u^3z^5-2u^2z^5+2u^3z^4+3u^2z^4+2uz^4-3u^2z^3-3uz^3-u^2z^2+3uz^2-z^2+uz+z-1}{(1-z)^4(1-zu)^4}$$
$$+\frac{1+u-2zu}{(1-z)(1-zu)}G(z,u).$$

Finally, for the distance to the identity permutation we get

$$\frac{\partial}{\partial z}G(z,u)=\frac{z^3u^4}{(1-zu)^5(1-z)}+\frac{u}{1-zu}G(z,u)+\frac{z^2u^3}{(1-zu)^4(1-z)}$$
$$+\frac{z^2u}{(1-z)^5}+\frac{z^3u}{(1-zu)^5(1-z)}+\frac{1}{1-z}G(z,u)$$

or

$$\frac{\partial}{\partial z}G(z,u)=\frac{1+u-2zu}{(1-zu)(1-z)}G(z,u)$$
$$+\ uz^2\frac{u^4z^4+u^2z^4-4u^3z^3-4u^2z^3+12u^2z^2-4u^2z-4uz+u^2+1}{(1-zu)^5(1-z)^5}.$$

Solve this to obtain

$$G(z,u)=\frac{1}{2}\frac{u}{(1-z)(1-zu)}\log\frac{1}{1-z}$$
$$+\frac{1}{2}\frac{1}{(1-z)(1-zu)}\log\frac{1}{1-zu}$$
$$+\frac{1}{4}zu\frac{3z^3u^2+3z^3u-2z^2u^2-12z^2u-2z^2+7zu+7z-4}{(1-zu)^3(1-z)^3}$$

for inversions;

$$G(z,u)=2\,\frac{u}{(1-zu)(1-z)}\log\frac{1}{1-z}+2\,\frac{1}{(1-zu)(1-z)}\log\frac{1}{1-zu}$$
$$-3\,\frac{zu}{(1-zu)(1-z)}$$

for fixed points;

$$G(z,u)=(1+H_m)\,\frac{u}{(1-zu)(1-z)}\log\frac{1}{1-z}+(1+H_m)\,\frac{1}{(1-zu)(1-z)}\log\frac{1}{1-zu}$$
$$-\left((1+2H_m)zu+\sum_{k=2}^{m}\frac{1}{k}\big(H_m-H_{k-1}\big)(uz^k+u^kz^k)\right)\,\frac{1}{(1-zu)(1-z)}$$

for cycles of length up to m;

$$G(z,u)=\frac{1}{(1-zu)(1-z)}\left(\frac{1}{2}u\log^2\frac{1}{1-z}+\frac{1}{2}\log^2\frac{1}{1-zu}\right)$$
$$-\frac{zu}{(1-zu)(1-z)}$$

$$+\frac{1}{(1-zu)(1-z)}\left(u\,\log\frac{1}{1-z}+\log\frac{1}{1-zu}\right)$$

for cycles;

$$G(z,u)=\frac{u}{(1-z)(1-zu)}\log\frac{1}{1-z}+\frac{1}{(1-z)(1-zu)}\log\frac{1}{1-zu}$$
$$+\frac{1}{2}zu\frac{-2z^4u^2+5z^3u^2+5z^3u-2z^2u^2-12z^2u-2z^2+5zu+5z-2}{(1-zu)^3(1-z)^3}$$

for the expected cycle length; and

$$G(z,u)=-\frac{1}{3}\,uz^3\,\frac{u^3z^3+u^2z^3-6u^2z^2+3u^2z+3uz-u^2-1}{(1-zu)^4(1-z)^4}$$

for the distance to the identity permutation.

4.1 Extracting coefficients

To obtain the desired expectations $E_{n,j}$ of the considered parameters, we have to extract coefficients from $G(z,u)$. Recall that

$$\frac{1}{1-z}\log\frac{1}{1-z}=\sum_{j\geq 1}H_jz^j,$$

with harmonic numbers $H_m=\sum_{1\leq k\leq m}1/k$. Hence

$$\sum_{n\geq 1}\sum_{j=1}^{n}H_ju^jz^n=\sum_{j\geq 1}H_ju^j\frac{z^j}{1-z}=\frac{1}{1-z}\frac{1}{1-zu}\log\frac{1}{1-zu}$$

and

$$\sum_{n\geq 1}\sum_{j=1}^{n}H_{n+1-j}u^jz^n=\sum_{n\geq 1}\sum_{j=1}^{n}H_ju^{n+1-j}z^n=\sum_{j\geq 1}H_jz^{j-1}\frac{zu}{1-zu}$$
$$=\frac{u}{1-zu}\frac{1}{1-z}\log\frac{1}{1-z}.$$

Note that

$$\sum_{n\geq 1}\sum_{j=1}^{n}j(j-5)u^jz^n=\sum_{j\geq 1}^{n}j(j-5)u^j\frac{z^j}{1-z}=\frac{1}{1-z}\sum_{j\geq 1}\big(j(j-1)-4j\big)(zu)^j$$
$$=\frac{1}{1-z}(zu)^2\frac{2}{(1-zu)^3}-\frac{1}{1-z}zu\frac{4}{(1-zu)^2}$$

and

$$\sum_{n\geq 1}\sum_{j=1}^{n}(n+1-j)(n-4-j)u^jz^n=\sum_{n\geq 1}\sum_{j=1}^{n}j(j-5)u^{n+1-j}z^n$$
$$=\sum_{j\geq 1}^{n}j(j-5)z^{j-1}\frac{zu}{1-zu}=\frac{zu}{1-zu}\sum_{j\geq 1}\big(j(j-1)-4j\big)z^{j-1}$$
$$=\frac{zu}{1-zu}z\frac{2}{(1-z)^3}-\frac{zu}{1-zu}\frac{4}{(1-z)^2}$$

and simplify

$$\frac{1}{1-z}(zu)^2\frac{2}{(1-zu)^3}-\frac{1}{1-z}zu\frac{4}{(1-zu)^2}+\frac{zu}{1-zu}z\frac{2}{(1-z)^3}-\frac{zu}{1-zu}\frac{4}{(1-z)^2}$$

to obtain

$$2zu\frac{-4+7zu-12z^2u-2z^2u^2+3z^3u+3z^3u^2+7z-2z^2}{(1-zu)^3(1-z)^3}.$$

This shows that

$$E_{n,j}=\frac{1}{8}(n+1-j)(n-4-j)+\frac{1}{8}j(j-5)+\frac{1}{2}H_{n+1-j}+\frac{1}{2}H_j$$

for inversions; and

$$E_{n,j}=2H_{n+1-j}+2H_j-3$$

for fixed points. We have

$$E_{n,j}=(1+H_m)(H_{n+1-j}+H_j)-\left(1+2H_m+2\sum_{k=2}^{m}\frac{1}{k}(H_m-H_{k-1})\right)$$

or

$$E_{n,j}=(1+H_m)(H_{n+1-j}+H_j)-\left(1+H_m^2+H_m^{(2)}\right)$$

for cycles of length up to m when $m-1<j<n-m+2$. Furthermore, we have

$$E_{n,j}=\frac{1}{2}H_{n+1-j}^2-\frac{1}{2}H_{n+1-j}^{(2)}+\frac{1}{2}H_j^2-\frac{1}{2}H_j^{(2)}-1+H_{n+1-j}+H_j$$

for cycles;

$$E_{n,j}=\frac{1}{n}\left(\frac{1}{4}(n+1-j)(n-j)+\frac{1}{4}j(j-1)-1+H_{n+1-j}+H_j\right)$$

for the expected cycle length; and

$$E_{n,j}=\frac{1}{18}\left(n^3-n\right)-\frac{1}{6}(n-1)j(n+1-j)$$

for the distance to the identity permutation.

4.2 Grand averages

The grand averages E_n can be obtained via

$$E_n=\frac{1}{n}[z^n]G(z,1)$$

for all considered parameters except the expected cycle length, where we have

$$E_n=\frac{1}{n^2}[z^n]G(z,1).$$

In the instance of inversions we get

$$G(z,1)=\frac{1}{(1-z)^2}\log\frac{1}{1-z}+\frac{1}{2}\frac{3z^2-2z}{(1-z)^4}$$

whence

$$[z^n]G(z,1)=(n+1)H_n+\frac{n(n^2-6n-19)}{12}$$

or

$$E_n=\left(1+\frac{1}{n}\right)H_n+\frac{n^2-6n-19}{12}.$$

In the instance of fixed points we get

$$G(z,1)=\frac{4}{(1-z)^2}\log\frac{1}{1-z}-\frac{3z}{(1-z)^2}$$

whence

$$[z^n]G(z,1) = 4(n+1)H_n - 7n$$

or

$$E_n = 4\left(1+\frac{1}{n}\right)H_n - 7.$$

In the instance of cycles up to some length m we get

$$\begin{aligned} G(z,1) = {} & 2(1+H_m)\frac{1}{(1-z)^2}\log\frac{1}{1-z} \\ & - \left((1+2H_m)z + 2\sum_{k=2}^{m}\frac{1}{k}(H_m - H_{k-1})z^k\right)\frac{1}{(1-z)^2} \end{aligned}$$

whence

$$\begin{aligned} [z^n]G(z,1) & = 2(1+H_m)H_n(n+1) - 2(1+H_m)n \\ & - \left((1+2H_m)n + 2\sum_{k=2}^{m}\frac{1}{k}(H_m - H_{k-1})(n+1-k)\right) \\ & = 2(1+H_m)H_n(n+1) - 2(1+H_m)n \\ & - \left((1+2H_m)n + (n+1)H_m^2 + (n+1)H_m^{(2)} - 2nH_m - 2m\right) \\ & = 2(1+H_m)H_n(n+1) - 2(1+H_m)n \\ & - \left((n+1)H_m^2 + (n+1)H_m^{(2)} + n - 2m\right) \end{aligned}$$

or

$$E_n = 2(1+H_m)H_n\left(1+\frac{1}{n}\right) - 2(1+H_m) - \left(1+\frac{1}{n}\right)H_m^2 - \left(1+\frac{1}{n}\right)H_m^{(2)} - 1 + 2\frac{m}{n}$$

for $n > m-2$. In the instance of cycles of arbitrary length we get

$$G(z,1) = \frac{1}{(1-z)^2}\log^2\frac{1}{1-z} - \frac{z}{(1-z)^2} + 2\frac{1}{(1-z)^2}\log\frac{1}{1-z}$$

whence

$$[z^n]G(z,1) = \left((H_{n+1}-1)^2 - (H_{n+1}^{(2)} - 1)\right)(n+1) - 3n + 2(n+1)H_n$$

or

$$E_n = \left(H_n^2 - H_n^{(2)}\right)\left(1+\frac{1}{n}\right) + \frac{2}{n}H_n - 1.$$

In the instance of the expected cycle length we get

$$G(z,1) = 2\frac{1}{(1-z)^2}\log\frac{1}{1-z} + \frac{-z^5+5z^4-8z^3+5z^2-z}{(1-z)^6} = 2\frac{1}{(1-z)^2}\log\frac{1}{1-z} + \frac{-z(z^2-3z+1)}{(1-z)^4}$$

whence

$$[z^n]G(z,1) = 2(n+1)H_n - 2n + \frac{n(n^2-7)}{6} = 2(n+1)H_n + \frac{n(n^2-19)}{6}$$

or

$$E_n = \frac{2}{n}\left(1+\frac{1}{n}\right)H_n + \frac{1}{6}n\left(1 - \frac{19}{n^2}\right).$$

Finally, for the distance to the identity permutation we get

$$G(z,1) = \frac{2}{3}\frac{z^3}{(1-z)^5}$$

whence

$$[z^n]G(z,1) = \frac{1}{36}(n+1)n(n-1)(n-2)$$

or

$$E_n = \frac{1}{36}(n+1)(n-1)(n-2).$$

4.3 Asymptotics

We need an effective means of comparing the disorder that is present in random permutations to the disorder that is present in a permutation after a single ordinary quickselect. Hence we compare the asymptotically dominant terms of $r_n'(1)$ (resp. $r_n'(1)/n$ in the case of the expected cycle length) and of E_n.

Recall that

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^4}\right)$$

and

$$H_n^{(2)} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right).$$

- **Inversions.** We have

$$r_n'(1) \sim \frac{1}{4}n^2 \quad \text{and} \quad E_n \sim \frac{1}{12}n^2.$$

- **Cycles of length less than or equal to some m.** Note that $r_n'(1) = H_m$ for $n \geq m$. The dominant term of E_n is $2(1 + H_m)H_n$. We have

$$r_n'(1) \sim H_m \quad \text{and} \quad E_n \sim 2(1 + H_m)\log n.$$

  For fixed points this becomes

$$r_n'(1) \sim 1 \quad \text{and} \quad E_n \sim 4\log n.$$

- **Cycles of any length.** Note that $r_n'(1) = H_n$. The dominant term of E_n is H_{n+1}^2. We have

$$r_n'(1) \sim \log n \quad \text{and} \quad E_n \sim \log^2 n.$$

- **Expected cycle length.** We have

$$\frac{r_n'(1)}{n} \sim \frac{1}{2}n \quad \text{and} \quad E_n \sim \frac{1}{6}n.$$

- **Distance to the identity permutation.** We have

$$r_n'(1) \sim \frac{1}{6}n^3 \quad \text{and} \quad E_n \sim \frac{1}{36}n^3.$$

5 Median-of-three pivoting

Translating recurrence relations (3c) and (3d) into a differential equation for the trivariate generating functions leads now to differential equations of higher order.

We have

$$\begin{aligned} n(n-1)(n-2)\ q_{n,j}(v)u^j z^{n-3} &= 6u^2 \sum_{p=1}^{j-1}(p-1)r_{p-1}(v)u^{p-2}z^{p-2}(n-p)q_{n-p,j-p}(v)u^{j-p}z^{n-p-1} \\ &+ 6u^2\ (j-1)r_{j-1}(v)u^{j-2}z^{j-2}(n-j)r_{n-j}(v)z^{n-j-1} \\ &+ 6\sum_{p=j+1}^{n}(p-1)q_{p-1,j}(v)u^j z^{p-2}(n-p)r_{n-p}(v)z^{n-p-1} \end{aligned}$$

for inversions and the distance to the identity permutation and

$$\begin{aligned} n(n-1)(n-2)\ q_{n,j}(v)u^j z^{n-3} &= 6u^2\ v\sum_{p=1}^{j-1}(p-1)r_{p-1}(v)u^{p-2}z^{p-2}(n-p)q_{n-p,j-p}(v)u^{j-p}z^{n-p-1} \\ &+ 6u^2\ v\ (j-1)r_{j-1}(v)u^{j-2}z^{j-2}(n-j)r_{n-j}(v)z^{n-j-1} \end{aligned}$$

$$+ 6\,v \sum_{p=j+1}^{n} (p-1) q_{p-1,j}(v) u^j z^{p-2} (n-p) r_{n-p}(v) z^{n-p-1}$$

for fixed points and cycles.

Let $R'(z,v) = \frac{\partial}{\partial z} R(z,v)$. Then

$$\frac{\partial^3}{\partial z^3} F(z,u,v) = 6u^2 R'(zu,v) \frac{\partial}{\partial z} F(z,u,v) + 6u^2 R'(zu,v) R'(z,v) + 6R'(z,v) \frac{\partial}{\partial z} F(z,u,v)$$

for inversions and the distance to the identity permutation and

$$\frac{\partial^3}{\partial z^3} F(z,u,v) = 6u^2 v R'(zu,v) \frac{\partial}{\partial z} F(z,u,v) + 6u^2 v R'(zu,v) R'(z,v) + 6vR'(z,v) \frac{\partial}{\partial z} F(z,u,v)$$

for fixed points, the expected cycle length and cycles.

Differentiate with respect to v to obtain

$$\begin{aligned} \frac{\partial}{\partial v} \frac{\partial^3}{\partial z^3} F(z,u,v) = 6u^2 \frac{\partial}{\partial v} R'(zu,v) \frac{\partial}{\partial z} F(z,u,v) + 6u^2 R'(zu,v) \frac{\partial}{\partial v} \frac{\partial}{\partial z} F(z,u,v) + 6u^2 \frac{\partial}{\partial v} R'(zu,v) R'(z,v) \\ + 6u^2 R'(zu,v) \frac{\partial}{\partial v} R'(z,v) + 6 \frac{\partial}{\partial v} R'(z,v) \frac{\partial}{\partial z} F(z,u,v) + 6R'(z,v) \frac{\partial}{\partial v} \frac{\partial}{\partial z} F(z,u,v) \end{aligned}$$

for inversions and the distance to the identity permutation and

$$\begin{aligned} \frac{\partial}{\partial v} \frac{\partial^3}{\partial z^3} F(z,u,v) = 6u^2 v R'(zu,v) \frac{\partial}{\partial z} F(z,u,v) + 6u^2 v \frac{\partial}{\partial v} R'(zu,v) \frac{\partial}{\partial z} F(z,u,v) + 6u^2 v R'(zu,v) \frac{\partial}{\partial v} \frac{\partial}{\partial z} F(z,u,v) \\ + 6u^2 v R'(zu,v) R'(z,v) + 6u^2 v \frac{\partial}{\partial v} R'(zu,v) R'(z,v) + 6u^2 v R'(zu,v) \frac{\partial}{\partial v} R'(z,v) \\ + 6vR'(z,v) \frac{\partial}{\partial z} F(z,u,v) + 6v \frac{\partial}{\partial v} R'(z,v) \frac{\partial}{\partial z} F(z,u,v) + 6R'(z,v) \frac{\partial}{\partial v} \frac{\partial}{\partial z} F(z,u,v) \end{aligned}$$

for fixed points, the expected cycle length and cycles.

Note that

$$\left.\frac{\partial}{\partial z} F(z,u,v)\right|_{v=1} = \frac{\partial}{\partial z} F(z,u,1) = \frac{u - u^2 z^2}{(1-z)^2 (1-zu)^2}$$

and

$$R'(z,v)|_{v=1} = \frac{\partial}{\partial z} \Big(R(z,v) \Big)\Big|_{v=1} = \frac{1}{(1-z)^2}.$$

We must evaluate

$$\left.\frac{\partial}{\partial v} R'(z,v)\right|_{v=1} = \frac{\partial}{\partial z} \left(\left.\frac{\partial}{\partial v} R(z,v)\right|_{v=1} \right)$$

for all our parameters:

- **Inversions.**
$$\frac{1}{2} \frac{z^2 + 2z}{(1-z)^4}.$$
- **Cycles of length less than or equal to some m.** We only treat the instance $m = 1$:
$$\frac{1}{(1-z)^2}.$$
- **Cycles of any length.**
$$\frac{1}{(1-z)^2} \left(1 + \log \frac{1}{1-z} \right).$$

- **Expected cycle length.**
$$\frac{1+2z}{(1-z)^4}$$

- **Distance to the identity permutation.**
$$\frac{2z^2+2z}{(1-z)^5}.$$

We evaluate now at $v=1$ and set
$$\Phi(z,u) = \frac{\partial}{\partial v}\frac{\partial}{\partial z}F(z,u,v)\bigg|_{v=1}.$$
This results in the second order differential equations
$$\frac{\partial^2}{\partial z^2}\Phi(z,u) = \frac{3zu^4(2+zu)(1-uz^2)}{(1-zu)^6(1-z)^2} + \frac{6u^2}{(1-zu)^2}\Phi(z,u) + \frac{3zu^3(2+zu)}{(1-zu)^4(1-z)^2} + \frac{3zu^2(2+z)}{(1-zu)^2(1-z)^4} + \frac{3zu(2+z)(1-z^2u)}{(1-zu)^2(1-z)^6} + \frac{6}{(1-z)^2}\Phi(z,u)$$
for inversions,
$$\frac{\partial^2}{\partial z^2}\Phi(z,u) = \frac{12u^3(1-z^2u)}{(1-zu)^4(1-z)^2} + \frac{6u^2}{(1-zu)^2}\Phi(z,u) + \frac{18u^2}{(1-zu)^2(1-z)^2} + \frac{12u(1-z^2u)}{(1-zu)^2(1-z)^4} + \frac{6}{(1-z)^2}\Phi(z,u)$$
for fixed points,
$$\frac{\partial^2}{\partial z^2}\Phi(z,u) = \frac{12u^3(1-z^2u)}{(1-zu)^4(1-z)^2} + \frac{6u^3(1-z^2u)}{(1-zu)^4(1-z)^2}\log\frac{1}{1-zu} + \frac{6u^2}{(1-zu)^2}\Phi(z,u) + \frac{18u^2}{(1-zu)^2(1-z)^2} + \frac{6u^2}{(1-zu)^2(1-z)^2}\log\frac{1}{1-zu} + \frac{6u^2}{(1-zu)^2(1-z)^2}\log\frac{1}{1-z} + \frac{12u(1-z^2u)}{(1-zu)^2(1-z)^4} + \frac{6u(1-z^2u)}{(1-zu)^2(1-z)^4}\log\frac{1}{1-z} + \frac{6}{(1-z)^2}\Phi(z,u)$$
for cycles,
$$\frac{\partial^2}{\partial z^2}\Phi(z,u) = \frac{6u^2(u-z^2u^2)}{(1-zu)^4(1-z)^2} + \frac{6u^2(1+2zu)(u-z^2u^2)}{(1-zu)^6(1-z)^2} + \frac{6u^2}{(1-zu)^2}\Phi(z,u) + \frac{6u^2}{(1-zu)^2(1-z)^2} + \frac{6u^2(1+2zu)}{(1-zu)^4(1-z)^2} + \frac{6u^2(1+2z)}{(1-zu)^2(1-z)^4} + \frac{6(u-z^2u^2)}{(1-zu)^2(1-z)^4} + \frac{6(1+2z)(u-z^2u^2)}{(1-z)^6(1-zu)^2} + \frac{6}{(1-z)^2}\Phi(z,u)$$
for the expected cycle length and
$$\frac{\partial^2}{\partial z^2}\Phi(z,u) = \frac{12zu^4(1+zu)(1-z^2u)}{(1-zu)^7(1-z)^2} + \frac{6u^2}{(1-zu)^2}\Phi(z,u) + \frac{12zu^4(1+zu)}{(1-zu)^5(1-z)^2} + \frac{12zu^2(1+z)}{(1-zu)^2(1-z)^5} + \frac{12zu(1+z)(1-z^2u)}{(1-zu)^2(1-z)^7} + \frac{6}{(1-z)^2}\Phi(z,u)$$
for the distance to the identity permutation.

The differential equation
$$\frac{\partial^2}{\partial z^2}\Phi(z,u) - 6\left(\frac{1}{(1-z)^2} + \frac{u^2}{(1-zu)^2}\right)\Phi(z,u) = g(z,u)$$

with different $g(z,u)$'s according to the parameter under consideration can be transformed into a Hypergeometric differential equation

$$t(1-t)G_{tt}(t,u) - 4(1-2t)G_t(t,u) - 8G(t,u) = \frac{t^3(1-t)^3(1-u)^6}{u^4} g\big(1 + \tfrac{1-u}{u}t, u\big).$$

There we used the substitutions

$$\Phi(z,u) = \frac{1}{(1-z)^2(1-zu)^2} E(z,u)\,,$$

$z = 1 + \frac{1-u}{u}t$ and $G(t,u) = E\big(1 + \frac{1-u}{u}t, u\big)$. This procedure was also used in [4].

The corresponding homogeneous differential equation has the solution

$$G^{\mathrm{hom}}(t,u) = k_1(u)(1-2t) + k_2(u)t^5\Big(1 - 2t + \frac{10}{7}t^2 - \frac{5}{14}t^3\Big).$$

Solving the differential equation by variation of the constants, backsubstitution and extraction of coefficients leads finally to the expectations[1]

$$\begin{aligned}
E_{n,j} &= -\frac{6}{35}H_n(3n^2 - 6jn - 3n + 4 - 6j + 6j^2) + \frac{9}{35}H_j(2j^2 - 6j + 5)\\
&\quad + \frac{9}{35}H_{n+1-j}(2n^2 - 2n - 4jn + 2j^2 + 2j + 1) - \frac{6}{7j} - \frac{6}{7(n+1-j)}\\
&\quad + \frac{3}{28}n^2 + \frac{18}{35}jn - \frac{167}{140}n + \frac{814}{245} - \frac{921}{2450}j^2 - \frac{153}{70}j\\
&\quad - \frac{3}{35}\frac{48j - 25 - 36j^2 + 3j^3}{n} + \frac{3}{35}\frac{(j^2 - 2j + 10)(j-1)^2}{n(n-1)} + \frac{1}{175}\frac{(j-1)(j-2)(2j-3)(3j^2 - 9j + 50)}{n(n-1)(n-2)}\\
&\quad + \frac{3}{175}\frac{(j-1)(j-3)(j^2 - 4j + 25)(j-2)^2}{n(n-1)(n-2)(n-3)} - \frac{3}{245}\frac{(j-1)(j-2)(j-3)(j-4)(2j-5)(j^2 - 5j + 35)}{n(n-1)(n-2)(n-3)(n-4)}\\
&\quad + \frac{1}{490}\frac{(j-1)(j-2)(j-4)(j-5)(3j^2 - 18j + 140)(j-3)^2}{n(n-1)(n-2)(n-3)(n-4)(n-5)} \quad \text{for } 5 \le j \le n-4,\\
E_{n,4} &= -\frac{87}{35}H_n + \frac{1}{4900}\frac{525n^5 - 4900n^4 - 131843n^2 + 48556n^3 + 104042n - 13440}{n(n-1)(n-2)} \quad \text{for } n \ge 8,\\
E_{4,4} &= \frac{1}{4},\ E_{5,4} = \frac{13}{20},\ E_{6,4} = \frac{17}{20},\ E_{7,4} = \frac{13}{10},\\
E_{n,3} &= -\frac{3}{7}H_n + \frac{1}{980}\frac{105n^4 - 770n^3 - 2090n + 2755n^2 - 588}{n(n-1)} \quad \text{for } n \ge 7,\ E_{3,3} = 0,\ E_{4,3} = \frac{1}{4},\ E_{5,3} = \frac{2}{5},\ E_{6,3} = \frac{17}{20},\\
E_{n,2} &= \frac{3}{28}n^2 - \frac{19}{28}n - \frac{1}{175} + \frac{3}{5}H_n \quad \text{for } n \ge 6,\ E_{2,2} = 0,\ E_{3,2} = 0,\ E_{4,2} = \frac{1}{4},\ E_{5,2} = \frac{13}{20},\\
E_{n,1} &= \frac{3}{28}n^2 - \frac{19}{28}n - \frac{1}{175} + \frac{3}{5}H_n \quad \text{for } n \ge 5,\ E_{1,1} = 0,\ E_{2,1} = 0,\ E_{3,1} = 0,\ E_{4,1} = \frac{1}{4}
\end{aligned}$$

for inversions,

$$\begin{aligned}
E_{n,j} &= \frac{48}{35}H_n + \frac{36}{35}H_j + \frac{36}{35}H_{n+1-j} + \frac{24}{35j} + \frac{24}{35(n+1-j)}\\
&\quad - \frac{583}{175} + \frac{12}{35}\frac{3j-5}{n} - \frac{24}{35}\frac{(j-1)^2}{n(n-1)} - \frac{8}{35}\frac{(j-1)(j-2)(2j-3)}{n(n-2)(n-1)} - \frac{12}{35}\frac{(j-1)(j-3)(j-2)^2}{n(n-2)(n-1)(n-3)}\\
&\quad + \frac{12}{35}\frac{(2j-5)(j-1)(j-2)(j-3)(j-4)}{n(n-2)(n-1)(n-3)(n-4)} - \frac{8}{35}\frac{(j-5)(j-1)(j-2)(j-4)(j-3)^2}{n(n-2)(n-1)(n-3)(n-4)(n-5)} \quad \text{for } 5 \le j \le n-4,\\
E_{n,4} &= \frac{12}{5}H_n - \frac{1}{175}\frac{-537n^2 + 179n^3 + 1618n - 1680}{n(n-2)(n-1)} \quad \text{for } n \ge 8,
\end{aligned}$$

[1] This technique is described in more detail in [7]. The paper [8] is also of relevance here.

$$E_{4,4} = \frac{7}{2},\ E_{5,4} = 4,\ E_{6,4} = \frac{23}{5},\ E_{7,4} = 5,$$

$$E_{n,3} = \frac{12}{5}H_n - \frac{1}{175}\frac{209n^2 - 209n + 420}{n(n-1)} \text{ for } n \geq 7,\ E_{3,3} = 3,\ E_{4,3} = \frac{7}{2},\ E_{5,3} = \frac{21}{5},\ E_{6,3} = \frac{23}{5},$$

$$E_{n,2} = -\frac{37}{25} + \frac{12}{5}H_n \text{ for } n \geq 6,\ E_{2,2} = 2,\ E_{3,2} = 3,\ E_{4,2} = \frac{7}{2},\ E_{5,2} = 4,$$

$$E_{n,1} = -\frac{37}{25} + \frac{12}{5}H_n \text{ for } n \geq 5,\ E_{1,1} = 1,\ E_{2,1} = 2,\ E_{3,1} = 3,\ E_{4,1} = \frac{7}{2}$$

for fixed points,

$$\begin{aligned}
E_{n,j} = {} & \frac{12}{35}H_nH_j + \frac{12}{35}H_nH_{n+1-j} + \left(\frac{34}{35} - \frac{12}{35j} - \frac{12}{35(n+1-j)}\right)H_n + \frac{9}{35}H_j^2 + \frac{9}{35}H_{n+1-j}^2 - \frac{12}{35}H_jH_{n+1-j} \\
& + \Bigg(\frac{12}{35j} + \frac{12}{35(n+1-j)} + \frac{103}{175} - \frac{12}{35}\,\frac{j}{n} - \frac{6}{35}\,\frac{j\,(j-1)}{n\,(n-1)} - \frac{4}{35}\,\frac{j\,(j-1)\,(j-2)}{n\,(n-1)\,(n-2)} - \frac{3}{35}\,\frac{j\,(j-1)\,(j-2)\,(j-3)}{n\,(n-1)\,(n-2)\,(n-3)} \\
& \quad + \frac{6}{35}\,\frac{j\,(j-1)\,(j-2)\,(j-3)\,(j-4)}{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)} - \frac{2}{35}\,\frac{j\,(j-1)\,(j-2)\,(j-3)\,(j-4)\,(j-5)}{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}\Bigg)H_j \\
& + \Bigg(\frac{12}{35j} + \frac{12}{35(n+1-j)} - \frac{2}{175} + \frac{6}{7}\,\frac{j-1}{n} - \frac{6}{35}\,\frac{(j-1)\,(j-2)}{n\,(n-1)} - \frac{4}{35}\,\frac{(j-1)\,(j-2)\,(j-3)}{n\,(n-1)\,(n-2)} \\
& \quad - \frac{3}{35}\,\frac{(j-1)\,(j-2)\,(j-3)\,(j-4)}{n\,(n-1)\,(n-2)\,(n-3)} + \frac{6}{35}\,\frac{(j-1)\,(j-2)\,(j-3)\,(j-4)\,(j-5)}{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)} \\
& \quad - \frac{2}{35}\,\frac{(j-1)\,(j-2)\,(j-3)\,(j-4)\,(j-5)\,(j-6)}{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}\Bigg)H_{n+1-j} \\
& - \frac{3}{5}H_j^{(2)} - \frac{3}{5}H_{n+1-j}^{(2)} + \frac{1}{35}\,\frac{17\,n+5}{(n+1)\,(n+1-j)} + \frac{1}{35}\,\frac{17\,n+5}{(n+1)\,j} - \frac{6221}{5250} - \frac{1}{175}\,\frac{2\,j+23}{n} - \frac{1}{35}\,\frac{(3\,j-7)\,(j-1)}{n\,(n-1)} \\
& + \frac{1}{105}\,\frac{(8\,j-9)\,(j-1)\,(j-2)}{n\,(n-1)\,(n-2)} + \frac{1}{35}\,\frac{(9\,j-28)\,(j-1)\,(j-2)\,(j-3)}{n\,(n-1)\,(n-2)\,(n-3)} \\
& - \frac{2}{175}\,\frac{(31\,j-90)\,(j-1)\,(j-2)\,(j-3)\,(j-4)}{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)} \\
& + \frac{62}{525}\,\frac{(j-1)\,(j-2)\,(j-3)^2\,(j-4)\,(j-5)}{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)} \text{ for } 5 \leq j \leq n-4,
\end{aligned}$$

$$\begin{aligned}
E_{n,4} = {} & \frac{3}{5}H_n^2 + \frac{6}{25}\frac{4\,n^3 - 12\,n^2 - 7\,n + 20}{n\,(n-1)\,(n-2)}H_n - \frac{3}{5}H_n^{(2)} \\
& + \frac{3}{3500}\frac{53760\,n + 701\,n^6 + 4353\,n^4 - 11200 + 24348\,n^3 - 66356\,n^2 - 4206\,n^5}{n^2\,(n-1)^2\,(n-2)^2} \text{ for } n \geq 8,
\end{aligned}$$

$$E_{4,4} = \frac{15}{4},\ E_{5,4} = \frac{89}{20},\ E_{6,4} = \frac{21}{4},\ E_{7,4} = \frac{59}{10},$$

$$E_{n,3} = \frac{3}{5}H_n^2 + \frac{6}{25}\frac{4\,n^2 - 4\,n - 5}{n\,(n-1)}H_n - \frac{3}{5}H_n^{(2)} + \frac{2}{875}\frac{-525 - 282\,n^3 - 454\,n^2 + 141\,n^4 + 1645\,n}{n^2\,(n-1)^2} \text{ for } n \geq 7,$$

$$E_{3,3} = 3,\ E_{4,3} = \frac{15}{4},\ E_{5,3} = \frac{23}{5},\ E_{6,3} = \frac{21}{4},$$

$$E_{n,2} = \frac{3}{5}H_n^2 + \frac{24}{25}H_n - \frac{3}{5}H_n^{(2)} + \frac{1}{125} \text{ for } n \geq 6,\ E_{2,2} = 2,\ E_{3,2} = 3,\ E_{4,2} = \frac{15}{4},\ E_{5,2} = \frac{89}{20},$$

$$E_{n,1} = \frac{3}{5}H_n^2 + \frac{24}{25}H_n - \frac{3}{5}H_n^{(2)} + \frac{1}{125} \text{ for } n \geq 5,\ E_{1,1} = 1,\ E_{2,1} = 2,\ E_{3,1} = 3,\ E_{4,1} = \frac{15}{4}$$

for cycles,

$$E_{n,j} = \frac{18}{35}\frac{2j^2 - 6j + 5}{n}H_j + \frac{18}{35}\frac{2n^2 - 4jn - 2n + 2j^2 + 2j + 1}{n}H_{n+1-j} - \frac{12}{35}\frac{3n^2 - 6jn - 3n + 6j^2 - 6j + 4}{n}H_n$$

$$
\begin{aligned}
&- \frac{12}{7}\frac{1}{jn} - \frac{12}{7}\frac{1}{(n+1-j)n} + \frac{3}{14}n + \frac{36}{35}j - \frac{97}{70} + \frac{1628}{245n} - \frac{921}{1225}\frac{j^2}{n} - \frac{153}{35}\frac{j}{n} - \frac{6}{35}\frac{48\,j - 25 - 36\,j^2 + 3\,j^3}{n^2}\\
&+ \frac{6}{35}\frac{(j^2-2\,j+10)(j-1)^2}{n^2(n-1)} + \frac{2}{175}\frac{(j-1)(j-2)(2j-3)(3j^2-9j+50)}{n^2(n-1)(n-2)}\\
&+ \frac{6}{175}\frac{(j-1)(j-3)(j^2-4j+25)(j-2)^2}{n^2(n-1)(n-2)(n-3)} - \frac{6}{245}\frac{(j-1)(j-2)(j-3)(j-4)(2j-5)(j^2-5j+35)}{n^2(n-1)(n-2)(n-3)(n-4)}\\
&+ \frac{1}{245}\frac{(j-1)(j-2)(j-4)(j-5)(3j^2-18j+140)(j-3)^2}{n^2(n-1)(n-2)(n-3)(n-4)(n-5)} \text{ for } 5 \le j \le n-4,
\end{aligned}
$$

$$
\begin{aligned}
E_{n,4} &= -\frac{174}{35}\frac{H_n}{n} + \frac{1}{2450}\frac{525n^5 - 2450n^4 + 41206n^3 - 126943n^2 + 104042n - 13440}{n^2(n-1)(n-2)} \text{ for } n \ge 8,\\
&E_{4,4} = \frac{9}{8},\ E_{5,4} = \frac{63}{50},\ E_{6,4} = \frac{77}{60},\ E_{7,4} = \frac{48}{35},\\
E_{n,3} &= -\frac{6}{7}\frac{H_n}{n} + \frac{1}{490}\frac{105n^4 - 280n^3 + 2265n^2 - 2090n - 588}{n^2(n-1)} \text{ for } n \ge 7,\\
&E_{3,3} = 1,\ E_{4,3} = \frac{9}{8},\ E_{5,3} = \frac{29}{25},\ E_{6,3} = \frac{77}{60},\\
E_{n,2} &= \frac{6}{5}\frac{H_n}{n} + \frac{3n}{14} - \frac{5}{14} - \frac{2}{175n} \text{ for } n \ge 6,\ E_{2,2} = 1,\ E_{3,2} = 1,\ E_{4,2} = \frac{9}{8},\ E_{5,2} = \frac{63}{50},\\
E_{n,1} &= \frac{6}{5}\frac{H_n}{n} + \frac{3n}{14} - \frac{5}{14} - \frac{2}{175n} \text{ for } n \ge 5,\ E_{1,1} = 1,\ E_{2,1} = 1,\ E_{3,1} = 1,\ E_{4,1} = \frac{9}{8}
\end{aligned}
$$

for the expected cycle length and

$$
\begin{aligned}
E_{n,j} &= \frac{6}{35}H_n(7n^3 - 21jn^2 - 9n^2 + 21j^2n + 2n + 18jn - 18 - 39j^2 + 39j) - \frac{6}{35}H_j(j-1)(j-2)(7j-9)\\
&- \frac{6}{35}H_{n+1-j}(7n-2-7j)(n-j)(n-1-j) - \frac{108}{35j} - \frac{108}{35(n+1-j)}\\
&+ \frac{1}{24}n^3 - \frac{6}{5}jn^2 + \frac{143}{140}n^2 - \frac{3419}{840}n + \frac{9}{4}j^2n + \frac{279}{140}jn - \frac{255}{28}j + \frac{70743}{4900} - \frac{21}{10}j^3 + \frac{57}{196}j^2\\
&+ \frac{3}{140}\frac{49j^4 - 218j^3 + 803j^2 - 850j + 360}{n}\\
&+ \frac{6}{35}\frac{(5j^2-10j+18)(j-1)^2}{n(n-1)} + \frac{6}{35}\frac{(j-1)(j-2)(2j-3)(j^2-3j+6)}{n(n-1)(n-2)}\\
&+ \frac{6}{35}\frac{(j-1)(j-3)(j^2-4j+9)(j-2)^2}{n(n-1)(n-2)(n-3)} - \frac{6}{245}\frac{(j-1)(j-2)(j-3)(j-4)(2j-5)(5j^2-25j+63)}{n(n-1)(n-2)(n-3)(n-4)}\\
&+ \frac{3}{245}\frac{(j-1)(j-2)(j-4)(j-5)(5j^2-30j+84)(j-3)^2}{n(n-1)(n-2)(n-3)(n-4)(n-5)} \text{ for } 5 \le j \le n-4,\\
E_{n,4} &= \frac{684}{35}H_n + \frac{1}{29400}\frac{1225n^4 - 5250n^3 - 127225n^2 - 711282n + 302400}{n} \text{ for } n \ge 8,\\
&E_{4,4} = \frac{1}{2},\ E_{5,4} = \frac{8}{5},\ E_{6,4} = 2,\ E_{7,4} = \frac{16}{5},\\
E_{n,3} &= \frac{1}{24}n^3 - \frac{5}{28}n^2 - \frac{223}{168}n + \frac{144}{35}H_n - \frac{13177}{4900} \text{ for } n \ge 7,\ E_{3,3} = 0,\ E_{4,3} = \frac{1}{2},\ E_{5,3} = \frac{4}{5},\ E_{6,3} = 2,\\
E_{n,2} &= -\frac{5}{28}n^2 + \frac{1}{24}n^3 + \frac{29}{168}n - \frac{1}{140} \text{ for } n \ge 6,\ E_{2,2} = 0,\ E_{3,2} = 0,\ E_{4,2} = \frac{1}{2},\ E_{5,2} = \frac{8}{5},\\
E_{n,1} &= -\frac{5}{28}n^2 + \frac{1}{24}n^3 + \frac{29}{168}n - \frac{1}{140} \text{ for } n \ge 5,\ E_{1,1} = 0,\ E_{2,1} = 0,\ E_{3,1} = 0,\ E_{4,1} = \frac{1}{2}
\end{aligned}
$$

for the distance to the identity permutation.

Due to the symmetry $E_{n,j} = E_{n+1-j}$, this parameter is fully described by the above values.

5.1 Grand averages

For the "grand averages" E_n we get for inversions

$$E_n = \left(\frac{6}{7} + \frac{6}{7n}\right) H_n + \frac{1}{12}n^2 - \frac{1}{2}n - \frac{793}{588} + \frac{9}{98n} \text{ for } n \geq 6,$$
$$E_1 = 0,\ E_2 = 0,\ E_3 = 0,\ E_4 = \frac{1}{4},\ E_5 = \frac{3}{5},$$

for fixed points

$$E_n = \left(\frac{24}{7} + \frac{24}{7n}\right) H_n - \frac{38}{49n} - \frac{255}{49} \text{ for } n \geq 6,$$
$$E_1 = 1,\ E_2 = 2,\ E_3 = 3,\ E_4 = \frac{7}{2},\ E_5 = \frac{101}{25},$$

for cycles

$$E_n = \left(\frac{6}{7} + \frac{6}{7n}\right) H_n^2 + \left(\frac{30}{49} + \frac{114}{49n}\right) H_n - \left(\frac{6}{7n} + \frac{6}{7}\right) H_n^{(2)} - \frac{65}{343n} - \frac{618}{343} \text{ for } n \geq 6,$$
$$E_1 = 1,\ E_2 = 2,\ E_3 = 3,\ E_4 = \frac{15}{4},\ E_5 = \frac{112}{25},$$

for the expected cycle length

$$E_n = \frac{1}{6}n - \frac{793}{294n} + \frac{9}{49}\frac{1}{n^2} + \frac{12}{7}\left(\frac{1}{n^2} + \frac{1}{n}\right) H_n \text{ for } n \geq 6,$$
$$E_1 = 1,\ E_2 = 1,\ E_3 = 1,\ E_4 = \frac{9}{8},\ E_5 = \frac{31}{25},$$

and for the distance to the identity permutation

$$E_n = \frac{(n+1)(14n^3 - 49n^2 + 14n + 36)}{525n} \text{ for } n \geq 6,$$
$$E_1 = 0,\ E_2 = 0,\ E_3 = 0,\ E_4 = \frac{1}{2},\ E_5 = \frac{36}{25}.$$

6 Variances

In this section we sketch how to compute the variances of our parameters. This is done by computing the second factorial moment, from which the variance is obtained by adding the expectation and subtracting the square of the expectation. It turns out that the explicit expressions that we obtain are already of daunting complexity. Henceforth, we refrain of doing such calculation for the more complicated instance of median–of–three partition.

Start with

$$\begin{aligned}\frac{\partial^2}{\partial v^2}\frac{\partial}{\partial z}F(z,u,v) &= u\frac{\partial^2}{\partial v^2}R(zu,v)F(z,u,v) + 2u\frac{\partial}{\partial v}R(zu,v)\frac{\partial}{\partial v}F(z,u,v) + uR(zu,v)\frac{\partial^2}{\partial v^2}F(z,u,v)\\ &+ u\frac{\partial^2}{\partial v^2}R(zu,v)R(z,v) + 2u\frac{\partial}{\partial v}R(zu,v)\frac{\partial}{\partial v}R(z,v) + uR(zu,v)\frac{\partial^2}{\partial v^2}R(z,v)\\ &+ \frac{\partial^2}{\partial v^2}R(z,v)F(z,u,v) + 2\frac{\partial}{\partial v}R(z,v)\frac{\partial}{\partial v}F(z,u,v) + R(z,v)\frac{\partial^2}{\partial v^2}F(z,u,v)\end{aligned}$$

for the inversions and the distance to the identity permutation and with

$$\begin{aligned}\frac{\partial^2}{\partial v^2}\frac{\partial}{\partial z}F(z,u,v) &= u\frac{\partial^2}{\partial v^2}R(zu,v)F(z,u,v) + 2u\frac{\partial}{\partial v}R(zu,v)\frac{\partial}{\partial v}F(z,u,v) + uR(zu,v)\frac{\partial^2}{\partial v^2}F(z,u,v)\\ &+ 2u\frac{\partial}{\partial v}R(zu,v)F(z,u,v) + 2uR(zu,v)\frac{\partial}{\partial v}F(z,u,v)\end{aligned}$$

$$
\begin{aligned}
&+ u\frac{\partial^2}{\partial v^2}R(zu,v)R(z,v) + 2u\frac{\partial}{\partial v}R(zu,v)\frac{\partial}{\partial v}R(z,v) + uR(zu,v)\frac{\partial^2}{\partial v^2}R(z,v)\\
&+ 2u\frac{\partial}{\partial v}R(zu,v)R(z,v) + 2uR(zu,v)\frac{\partial}{\partial v}R(z,v)\\
&+ \frac{\partial^2}{\partial v^2}R(z,v)F(z,u,v) + 2\frac{\partial}{\partial v}R(z,v)\frac{\partial}{\partial v}F(z,u,v) + R(z,v)\frac{\partial^2}{\partial v^2}F(z,u,v)\\
&+ 2\frac{\partial}{\partial v}R(z,v)F(z,u,v) + 2R(z,v)\frac{\partial}{\partial v}F(z,u,v)
\end{aligned}
$$

for fixed points, the expected cycle length and cycles.

We finally get for the second factorial moments $M_{n,j}^{(2)}$:

Inversions.

$$
\begin{aligned}
M_{n,j}^{(2)} &= -\frac{1}{4}H_j^{(2)} - \frac{1}{4}H_{n+1-j}^{(2)}\\
&- \frac{1}{24}\frac{H_n\,(n+1)\left(n^3j - 4n^2j^2 + 3n^2j + 6j^3n - 8j^2n - 10jn + 9j^2 - 3j^4 - 12 + 6j^3 - 12j\right)}{j\,(n+1-j)}\\
&+ \frac{1}{4}H_j^2 + \frac{1}{4}H_{n+1-j}^2 + \frac{1}{2}H_{n+1-j}H_j\\
&+ \frac{1}{24}\frac{H_j\left(3n^3j - 12n^2j - 9n^2j^2 + 12j^3n - 4j^2n + j^4n - 12n - 43jn + 15j^2 - 12 - j^5 - 28j - 5j^4 + 19j^3\right)}{j\,(n+1-j)}\\
&+ \frac{1}{24}\frac{H_{n+1-j}}{j\,(n+1-j)}\left(n^4j + 7n^3j - 4n^3j^2 - 21n^2j^2 - 13n^2j + 6n^2j^3 - 12n - 53jn + 5j^2n + 24j^3n - 4j^4n\right.\\
&\left.-10j^4 - 34j + 32j^2 + 11j^3 + j^5 - 12\right)\\
&+ \frac{1}{3456}\frac{1}{j\,(n+1-j)}\left(1626j^4 + 702j^5 - 4422j^3 - 4296j^2 + 6624j - 234j^6 + 1728 + 702j^5n + 63jn^5 + 9382jn\right.\\
&-805n^3j + 3783n^2j^2 + 2191n^2j - 5094j^3n - 834j^4n + 2468j^2n - 175n^4j + 304n^3j^2 - 954j^4n^2 + 66n^2j^3\\
&\left.-315n^4j^2 + 738n^3j^3\right) \text{ for } n \ge j \ge 1.
\end{aligned}
$$

Fixed points.

$$
\begin{aligned}
M_{n,j}^{(2)} &= -4H_j^{(2)} - 4H_{n+1-j}^{(2)} + 8\frac{H_n\,(n+1)}{(n+1-j)\,j} + 4H_j^2 + 4H_{n+1-j}^2 + 8H_{n+1-j}H_j\\
&- \frac{H_j\left(-9j^2 + 9jn + 9j + 8n + 8\right)}{(n+1-j)\,j} - \frac{H_{n+1-j}\left(-9j^2 + 9jn + 9j + 8n + 8\right)}{(n+1-j)\,j}\\
&+ \frac{1}{3}\frac{118j + 23n^2j^2 - 46j^3 + 47n^2j - 46j^3n - j^2n + 23j^4 + 60n - 95j^2 + 72 + 165jn + 12n^2}{j\,(n+2-j)\,(j+1)\,(n+1-j)} \text{ for } n-1 \ge j \ge 2,\\
M_{n,1}^{(2)} &= 4H_n^2 - 4H_n^{(2)} - H_n + \frac{4}{n} - \frac{5}{2} \text{ for } n \ge 2,\ M_{1,1}^{(2)} = 0.
\end{aligned}
$$

Cycles.

$$
\begin{aligned}
M_{n,j}^{(2)} &= \left(\frac{2}{(n+1-j)^2} - \frac{2}{n+1-j} - 2\frac{H_{n+1-j}}{n+1-j}\right)\sum_{k=1}^{j}\frac{H_{n-j+k}}{k} + \left(\frac{2}{j^2} - \frac{2}{j} - 2\frac{H_j}{j}\right)\sum_{k=1}^{n+1-j}\frac{H_{k+j-1}}{k}\\
&+ 2H_{n+1-j}H_{n+1-j}^{(3)} + 2H_jH_j^{(3)} - 4H_jH_j^{(2)} - 4H_{n+1-j}H_{n+1-j}^{(2)} - \frac{3}{2}H_{n+1-j}^{(4)} - \frac{3}{2}H_j^{(4)} + \frac{8}{3}H_{n+1-j}^{(3)} + \frac{8}{3}H_j^{(3)}\\
&+ \frac{3}{4}\left(H_j^{(2)}\right)^2 + \frac{3}{4}\left(H_{n+1-j}^{(2)}\right)^2 - \frac{3}{2}H_j^2H_j^{(2)} - \frac{3}{2}H_{n+1-j}^2H_{n+1-j}^{(2)} + \frac{1}{4}H_j^4 + \frac{1}{4}H_{n+1-j}^4 + \frac{4}{3}H_j^3 + \frac{4}{3}H_{n+1-j}^3\\
&+ \frac{1}{2}H_j^2H_{n+1-j}^2 + \frac{1}{2}H_j^{(2)}H_{n+1-j}^{(2)} - \frac{1}{2}H_j^2H_{n+1-j}^{(2)} - \frac{1}{2}H_{n+1-j}^2H_j^{(2)} + \left(\frac{1}{j} - 1\right)H_jH_{n+1-j}^{(2)} + \left(\frac{1}{n+1-j} - 1\right)H_{n+1-j}H_j^{(2)}\\
&+ \left(-1 + \frac{1}{j} - \frac{1}{j^2}\right)H_{n+1-j}^{(2)} + \left(-1 + \frac{1}{n+1-j} - \frac{1}{(n+1-j)^2}\right)H_j^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(1-\frac{1}{j}\right) H_j H_{n+1-j}^2+\left(1-\frac{1}{n+1-j}\right) H_{n+1-j} H_j^2 \\
&+\left(-\frac{2}{j^2}-\frac{2}{j}+\frac{1}{(n+1-j)^2}-\frac{1}{n+1-j}+1\right) H_j^2+\left(-\frac{2}{(n+1-j)^2}-\frac{2}{n+1-j}+\frac{1}{j^2}-\frac{1}{j}+1\right) H_{n+1-j}^2 \\
&+\left(-2-\frac{2(2n+1)}{(n+1)(n+1-j)}-\frac{2}{(n+1-j)^2(n+1)}+\frac{2}{(n+1-j)^3}-\frac{2(2n+1)}{(n+1) j}+\frac{2}{(n+1) j^2}+\frac{2}{j^3}\right) H_j \\
&+\left(-2-\frac{2(2n+1)}{(n+1)(n+1-j)}+\frac{2}{(n+1-j)^2(n+1)}+\frac{2}{(n+1-j)^3}-\frac{2(2n+1)}{(n+1) j}-\frac{2}{(n+1) j^2}+\frac{2}{j^3}\right) H_{n+1-j} \\
&+\left(2-\frac{2n}{(n+1)(n+1-j)}-\frac{2}{(n+1-j)^2}-\frac{2n}{(n+1) j}-\frac{2}{j^2}\right) H_j H_{n+1-j} \\
&+\left(\frac{2}{j^2}+\frac{2}{j}-\frac{2}{(n+1-j)^2}+\frac{2}{n+1-j}\right) H_n H_j+\left(\frac{2}{(n+1-j)^2}+\frac{2}{n+1-j}-\frac{2}{j^2}+\frac{2}{j}\right) H_n H_{n+1-j} \\
&+\left(\frac{2}{j}+\frac{2}{n+1-j}\right) H_n H_j H_{n+1-j}+\left(-\frac{2}{j^3}+\frac{4}{j}-\frac{2}{(n+1-j)^3}+\frac{4}{n+1-j}\right) H_n \\
&-\frac{2(2n-1)}{(n+1) j}-\frac{2(2n-1)}{(n+1)(n+1-j)}-\frac{2}{(n+1) j^3}-\frac{2}{(n+1)(n+1-j)^3}+12 \text{ for } n \geq j \geq 1.
\end{aligned}
$$

Expected cycle length.

$$
\begin{aligned}
M_{n,j}^{(2)} = &\frac{H_j^2}{n^2}+\frac{H_{n+1-j}^2}{n^2}-\frac{H_j^{(2)}}{n^2}-\frac{H_{n+1-j}^{(2)}}{n^2} \\
&-\left(\frac{1}{6} \frac{n^3-3jn^2+3n^2+3j^2n-6jn-10n+3j^2-3j-12}{n^2}-\frac{2}{jn^2}-\frac{2}{n^2(n+1-j)}\right) H_n \\
&+\left(\frac{1}{6} \frac{3n^2-6jn-9n+j^3+6j^2-13j-18}{n^2}-\frac{2}{jn^2}-\frac{2}{n^2(n+1-j)}\right) H_j \\
&+\left(\frac{1}{6} \frac{n^3-3jn^2+6n^2+3j^2n-12jn-13n-j^3+9j^2-2j-24}{n^2}-\frac{2}{jn^2}-\frac{2}{n^2(n+1-j)}\right) H_{n+1-j}+2 \frac{H_j H_{n+1-j}}{n^2} \\
&+\frac{1}{72} \frac{6n^4-24jn^3-2n^3+45j^2n^2-9jn^2-72n^2-42j^3n+27j^2n+147jn+68n+21j^4-42j^3-111j^2+132j+432}{n^2} \\
&+\frac{2}{jn^2(n+1)}+\frac{2}{n^2(n+1-j)(n+1)} \text{ for } n \geq j \geq 1.
\end{aligned}
$$

Distance to the identity permutation.

$$
\begin{aligned}
M_{n,j}^{(2)} = &\frac{2}{45} H_n (n+1)\left(n^4-5n^3j+4n^3+10n^2j^2+n^2-15n^2j+20j^2n-10j^3n-6n+10j-5j^2+5j^4-10j^3\right) \\
&-\frac{2}{45} j H_j (j-1)(j-2)(j+2)(j+1)-\frac{2}{45} H_{n+1-j}(n+1-j)(n-j)(n-1-j)(n+3-j)(n+2-j) \\
&+\frac{31}{6480} n^6-\frac{1}{2160} n^5(-27+62j)+\frac{1}{6480} n^4\left(343+465j^2-693j\right)-\frac{1}{6480} n^3\left(2588j+580j^3-1986j^2-687\right) \\
&+\frac{1}{6480} n^2\left(202+345j^4-2466j^3+5562j^2-2349j\right)-\frac{1}{6480} n\left(4628j^3-1053j^4+66j^5-236j+1344-3909j^2\right) \\
&+\frac{1}{3240} j(j-1)\left(11j^4-22j^3+691j^2-680j-252\right) \text{ for } n \geq j \geq 1.
\end{aligned}
$$

6.1 The variance of the mean

Of particular interest is the variance of the mean V_n for the considered parameters. In principle, this parameter can be obtained by summing up above obtained values, but it is easier to evaluate at $u = 1$ at the level of generating functions and extract coefficients.

We then get for inversions

$$V_n = \frac{n^4}{720} + \frac{5n^3}{432} - \frac{11n^2}{216} + \frac{247n}{432} + \frac{4607}{2160} + \left(-\frac{1}{n} - \frac{1}{n^2}\right) H_n^2 + \left(-\frac{n}{6} - \frac{1}{3} + \frac{11}{6n}\right) H_n + \left(-1 - \frac{1}{n}\right) H_n^{(2)} \text{ for } n \geq 1,$$

for fixed points

$$V_n = \left(-16 - \frac{16}{n}\right) H_n^{(2)} + \left(-\frac{16}{n} - \frac{16}{n^2}\right) H_n^2 + \left(\frac{58}{n} + 10\right) H_n - \frac{19}{3n} + \frac{5}{3} \text{ for } n \geq 2,\ V_1 = 0,$$

for cycles

$$\begin{aligned} V_n = &\left(-4 - \frac{2}{n} + \frac{2}{n^2}\right) H_n^2 H_n^{(2)} + \left(\frac{2}{3} - \frac{4}{n^2} + \frac{2}{3n}\right) H_n^3 + \left(-\frac{1}{n^2} - \frac{1}{n}\right) H_n^4 + \left(-\frac{1}{n^2} + 2 + \frac{1}{n}\right) \left(H_n^{(2)}\right)^2 \\ &+ \left(\frac{5}{n} + 5 - \frac{4}{n^2}\right) H_n^2 + \left(\frac{4}{3} + \frac{28}{3n}\right) H_n^{(3)} + \left(-8 - \frac{1}{6n}\right) H_n + \left(-5 - \frac{5}{n}\right) H_n^{(2)} + \left(-6 - \frac{6}{n}\right) H_n^{(4)} \\ &+ \left(\frac{4}{n^2} - 2 - \frac{10}{n}\right) H_n H_n^{(2)} + \left(\frac{8}{n} + 8\right) H_n H_n^{(3)} + 18 \text{ for } n \geq 1, \end{aligned}$$

for the expected cycle length

$$V_n = \frac{1}{720} \frac{7n^4 + 25n^3 - 185n^2 + 1895n + 6898}{n^2} - \frac{4(n+1)}{n^4} H_n^2 - \frac{4(n+1)}{n^3} H_n^{(2)} - \frac{2}{3} \frac{(n+5)(n-2)}{n^3} H_n \text{ for } n \geq 1,$$

and for the distance to the identity permutation

$$V_n = \frac{(n-1)(n-2)(n+1)(145\,n^3 + 836\,n^2 + 53\,n - 398)}{226800} \text{ for } n \geq 1.$$

7 Conclusion

In the following table we collect our basic findings. Note for instance that the number of cycles is *increasing*, since we are "closer" to the identity permutation, which has the most number of cycles.

	Random permutation	After quickselect	After quickselect (median–of–3)	Variance
Inversions	$\frac{n^2}{4}$	$\frac{n^2}{12}$	$\frac{n^2}{12}$	$\frac{n^4}{720}$
Fixed points	1	$4 \log n$	$\frac{24}{7} \log n$	$10 \log n$
Cycles	$\log n$	$\log^2 n$	$\frac{6}{7} \log^2 n$	$\frac{2}{3} \log^3 n$
Expected cycle length	$\frac{n}{2}$	$\frac{n}{6}$	$\frac{n}{6}$	$\frac{7n^2}{720}$
Distance to the identity permutation	$\frac{n^3}{6}$	$\frac{n^3}{36}$	$\frac{2n^3}{75}$	$\frac{29n^6}{45360}$

Table 1: Averages and variances with/without one round of quickselect. (leading term only)

At first glance it might seem surprising that the parameters "number of fixed points" and "number of cycles" are smaller after the median-of-three algorithm than after the ordinary Quickselect algorithm, what means that for these statistics the permutations are on average more disordered in the median-of-three case. The reason for this is that with median-of-three partition the number of recursive calls in the algorithm decreases and thus the requested element can be found "faster." Because every recursive call places one pivot element in the correct position and on average in every segment between the pivots we have one additional fixed point (these segments are random permutations), one expects that the average number of fixed points will asymptotically behave like twice the average number of recursive calls in Quickselect, a parameter that was studied in [4]. With the heuristic that for large n almost no pivots are neighbors, we get that asymptotically the number of fixed points is twice the number of pivots (or recursive calls). On average we make in the median-of-three case asymptotically 1/7 fewer recursive calls and thus we have about 1/7 fewer fixed points.

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