Mellin transforms and the functional equation of the Riemann Zeta function

Marko R. Riedel

July 8, 2022

We collect several examples of using Mellin transforms and the functional equation of the Riemann Zeta function to evaluate harmonic sums. These are extracted from my posts at math.stackexchange.com and have retained the question answer format used there.

The methods here are those of the papers by Szpankowski [Szp01] and Flajolet [FS96]. Similar techniques are used in [Vep06].

Contents

1 Evaluating Fourier series

Suppose we seek to show that

$$\sum_{n=1,3,5,\ldots} \frac{\sin^2(n/q)}{n^2} = \frac{\pi}{4q}.$$

As suggested we use

$$\sin^2 t = \frac{1-\cos(2t)}{2}$$

to get

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^2} \frac{1-\cos(2n/q)}{2}$$

which is

$$\frac{1}{2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos(2n/q).$$

We will be using

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^s} = \left(1 - \frac{1}{2^s}\right) \zeta(s)$$

which gives for the sum

$$\frac{1}{2}\left(1 - \frac{1}{2^2}\right)\zeta(2) - \frac{1}{2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos(2n/q)$$

or

$$\frac{\pi^2}{16} - \frac{1}{2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos(2n/q).$$

Introduce $S(x)$ given by

$$S(x) = \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos(nx) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)x)$$

so that we are interested in $S(2/q)$.

The sum term is harmonic and may be evaluated by inverting its Mellin transform.

Recall the harmonic sum identity

$$\mathfrak{M}\left(\sum_{k\ge 1} \lambda_k g(\mu_k x); s\right) = \left(\sum_{k\ge 1} \frac{\lambda_k}{\mu_k^s}\right) g^*(s)$$

where $g^*(s)$ is the Mellin transform of $g(x)$.

In the present case we have

$$\lambda_k = \frac{1}{(2k-1)^2}, \quad \mu_k = (2k-1) \quad \text{and} \quad g(x) = \cos(x).$$

We need the Mellin transform $g^*(s)$ of $g(x)$.

Now the Mellin transform of $\cos(x)$ was computed at math.stackexchange.com question 479586 and found to be

$$\Gamma(s)\cos(\pi s/2)$$

It follows that the Mellin transform $Q(s)$ of the harmonic sum $S(x)$ is given by

$$Q(s) = \Gamma(s)\cos(\pi s/2)\left(1 - \frac{1}{2^{s+2}}\right)\zeta(s+2)$$
$$\text{because} \quad \sum_{k\geq 1}\frac{\lambda_k}{\mu_k^s} = \sum_{k\geq 1}\frac{1}{(2k-1)^2}\frac{1}{(2k-1)^s} = \left(1 - \frac{1}{2^{s+2}}\right)\zeta(s+2)$$

for $\Re(s) > -1$.

The Mellin inversion integral here is

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} Q(s)/x^s ds$$

which we evaluate by shifting it to the left for an expansion about zero (the abscissa $\Re(s) = 1/2$ is in the intersection of $\langle -1, \infty\rangle$ and $\langle 0, 1\rangle$ from the cosine transform).

The zeros of the cosine term at the negative odd integers cancel the poles of the gamma function at those values. Additional cancelation is gained from the trivial zeros of the zeta function term $\zeta(s+2)$ at the even negative integers p with $p \leq -4$.

This leaves just two poles at $s = 0$ and $s = 1$ and we have

$$\text{Res}_{s=0} Q(s)/x^s = \frac{\pi^2}{8} \quad \text{and} \quad \text{Res}_{s=-1} Q(s)/x^s = -\frac{\pi}{4}x$$

and therefore

$$S(x) \sim \frac{\pi^2}{8} - \frac{\pi}{4}x.$$

We will see that this is exact for $x \in [0, \pi)$.

With $q \geq 1$ we have $2/q \leq 2$ and we get for the initial sum the form

$$\frac{\pi^2}{16} - \frac{1}{2}\frac{\pi^2}{8} + \frac{1}{2}\frac{\pi}{4}\frac{2}{q} = \frac{\pi}{4q}$$

which is the claim we were trying to prove.

We still need to prove exactness on $[0, \pi)$ to complete the argument.

Put $s = \sigma + it$ with $\sigma \leq -3/2$ where we seek to evaluate

$$\frac{1}{2\pi i}\int_{-3/2-i\infty}^{-3/2+i\infty} Q(s)/x^s ds$$

by shifting it to the left.

Recall that with $\sigma > 1$ and for $|t| \to \infty$ we have

$$|\zeta(\sigma + it)| \in \mathcal{O}(1).$$

Furthermore recall the functional equation of the Riemann Zeta function

$$\zeta(1-s) = \frac{2}{2^s \pi^s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(s)$$

which we re-parameterize like so

$$\zeta(s+2) = 2 \times (2\pi)^{s+1} \cos\left(-\frac{\pi(s+1)}{2}\right) \Gamma(-s-1)\zeta(-s-1)$$

which is

$$\zeta(s+2) = -2 \times (2\pi)^{s+1} \sin(\pi s/2) \frac{\Gamma(1-s)}{s(s+1)} \zeta(-s-1).$$

Substitute this into $Q(s)$ to obtain

$$\Gamma(s) \cos(\pi s/2) \left(1 - \frac{1}{2^{s+2}}\right) \times -2 \times (2\pi)^{s+1} \sin(\pi s/2) \frac{\Gamma(1-s)}{s(s+1)} \zeta(-s-1).$$

Use the reflection formula for the Gamma function to obtain

$$\cos(\pi s/2) \left(1 - \frac{1}{2^{s+2}}\right) \times -2 \times (2\pi)^{s+1} \sin(\pi s/2) \times \frac{\pi}{\sin(\pi s)} \frac{1}{s(s+1)} \zeta(-s-1),$$

in other words we have

$$Q(s) = -\pi(2\pi)^{s+1} \left(1 - \frac{1}{2^{s+2}}\right) \frac{\zeta(-s-1)}{s(s+1)}.$$

There are two components here, call them $Q_1(s)$ and $Q_2(s)$, which are

$$-\pi(2\pi)^{s+1} \frac{\zeta(-s-1)}{s(s+1)} \quad \text{and} \quad \pi(2\pi)^{s+1} \frac{1}{2^{s+2}} \frac{\zeta(-s-1)}{s(s+1)}.$$

We evaluate these with $\sigma < -5/2$. For the first component this implies (with $\sigma = -5/2$ we have $\Re(-s-1) = 3/2$)

$$|Q_1(s)/x^s| \sim 2\pi^2 (2\pi)^\sigma x^{-\sigma} |t|^{-2}.$$

or

$$|Q_1(s)/x^s| \sim 2\pi^2 (x/2/\pi)^{-\sigma} |t|^{-2}.$$

We see from the term in $|t|$ that the integral obviously converges. (This much we knew already.) Moreover, when $x \in (0, 2\pi)$ we have $(x/2/\pi)^{-\sigma} \to 0$ as $\sigma \to -\infty$. The term in x does not depend on the variable t of the integral and may be brought to the front. This means that the contribution from the left side of the rectangular contour that we employ as we shift to the left vanishes in the limit.

For the second component we get

$$|Q_2(s)/x^s| \sim \frac{\pi^2}{2}(\pi)^\sigma x^{-\sigma}|t|^{-2}.$$

or

$$|Q_2(s)/x^s| \sim \frac{\pi^2}{2}(x/\pi)^{-\sigma}|t|^{-2}.$$

This is the same as the first only now we have convergence in $(0, \pi)$.

Joining the bounds for $Q_1(s)$ and $Q_2(s)$ we have proved the exactness of the formula for $S(x)$ in the interval $(0, \pi)$ obtained earlier.

As I have mentioned elsewhere there is a theorem hiding here, namely that certain Fourier series can be evaluated by inverting their Mellin transforms which is not terribly surprising and which the reader is invited to state and prove.

This is math.stackexchange.com question 1153068.

2 Approximating a simple product

Let

$$P = \prod_{n\ge 1}\left(1 - \frac{1}{2^n}\right).$$

Introduce

$$S = \log P = \sum_{n\ge 1}\log\left(1 - \frac{1}{2^n}\right).$$

Now recall the harmonic sum identity for Mellin transforms:

$$\mathfrak{M}\left(\sum_{k\ge 1}\lambda_k g(\mu_k x); s\right) = \left(\sum_{k\ge 1}\frac{\lambda_k}{\mu_k^s}\right) g^*(s)$$

where $g^*(s)$ is the Mellin transform of $g(x)$.

Introducing

$$S(x) = \sum_{n\ge 1}\log\left(1 - \frac{1}{2^{nx}}\right)$$

so that $S = S(1)$ we see that $S(x)$ is harmonic with parameters

$$\lambda_k = 1, \quad \mu_k = k \quad \text{and} \quad g(x) = \log\left(1 - \frac{1}{2^x}\right)$$

and may be evaluated (approximated) by inverting its Mellin transform.

Now

$$\mathfrak{M}\left(\log\left(1-\frac{1}{2^x}\right); s\right) = \int_0^\infty \log\left(1-\frac{1}{2^x}\right) x^{s-1} dx = -\int_0^\infty \sum_{q\ge 1} \frac{2^{-qx}}{q} x^{s-1} dx.$$

This is

$$-\frac{\Gamma(s)}{(\log 2)^s} \sum_{q\ge 1} \frac{1}{q}\frac{1}{q^s} = -\frac{\Gamma(s)}{(\log 2)^s}\zeta(s+1).$$

It follows that the Mellin transform of $S(x)$ is

$$Q(s) = \mathfrak{M}(S(x); s) = -\frac{1}{(\log 2)^s}\Gamma(s)\zeta(s)\zeta(s+1).$$

Now perform Mellin inversion with the inversion integral being

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Q(s)/x^s ds$$

and shifting the integral to the left for an expansion about zero. There are only three singularities to consider because the trivial zeros of the two zeta function terms cancel the poles of the gamma function.

We have

$$\mathrm{Res}(Q(s)/x^s; s=1) = -1/6\,\frac{\pi^2}{\log 2 \times x},$$

$$\mathrm{Res}(Q(s)/x^s; s=0) = 1/2\,\log(2\,\pi) - 1/2\,\log\log 2 - 1/2\,\log x$$

and

$$\mathrm{Res}(Q(s)/x^s; s=-1) = 1/24\,\log 2 \times x.$$

Putting $x=1$ we obtain that

$$S(1) = S \approx -1/6\,\frac{\pi^2}{\log 2} + 1/2\,\log(2\,\pi) - 1/2\,\log\log 2 + 1/24\,\log 2.$$

This approximation is excellent (good to 24 digits) but not quite exact. E.g. setting $x = 1/2$ which is closer to zero we get 50 good digits, setting $x = 1/5$ we get 123 good digits and so on.

The conclusion is that

$$P \approx e^{-\pi^2/6/\log 2} \times \sqrt{2\pi} \times \frac{2^{1/24}}{\sqrt{\log 2}} \approx 0.28878809508660242127889 97.$$

Adddendum, July 2022. We can actually do a somewhat better and compute a functional equation for $S(x)$. This requires an evaluation of the remainder, which is

$$\frac{1}{2\pi i}\int_{-3/2-i\infty}^{-3/2+i\infty} Q(s)/x^s\; ds.$$

Substitute $s = -t$ in this integral to get (we get one minus from $Q(s)$, another from the differential, and a third reversing the direction of the line)

$$-\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}(\log 2)^t\Gamma(-t)\zeta(-t)\zeta(1-t)x^t dt$$

In view of the desired functional equation we now use the functional equation of the Riemann zeta function on $Q(s)$ to prove that the integrand of the last integral is in fact a scaled version of $Q(t)$.

Start with the functional equation

$$\zeta(1-s) = \frac{2}{2^s\pi^s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

and substitute this into $Q(s)$ to obtain

$$Q(s) = -\frac{1}{(\log 2)^s}\frac{\zeta(1-s)2^s\pi^s}{2\cos\left(\frac{\pi s}{2}\right)}\zeta(s+1).$$

Apply the functional equation again (this time to $\zeta(s+1)$) to get

$$-\frac{1}{(\log 2)^s}\frac{\zeta(1-s)2^s\pi^s}{2\cos\left(\frac{\pi s}{2}\right)}2^{s+1}\pi^s\cos\left(-\frac{\pi s}{2}\right)\Gamma(-s)\zeta(-s).$$

This is

$$Q(s) = -2^{2s}\pi^{2s}\frac{1}{(\log 2)^s}\Gamma(-s)\zeta(-s)\zeta(1-s).$$

We thus have for the remainder integral (there was a minus in front and the multiple of $Q(t)$ brings another one)

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}(\log 2)^t\frac{Q(t)}{2^{2t}\pi^{2t}}(\log 2)^t x^t dt$$
$$= \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}Q(t)\left(\frac{4\pi^2}{x(\log 2)^2}\right)^{-t} dt.$$

Together with the residues we have established the following functional equation

$$S(x) = -\frac{1}{6}\frac{\pi^2}{\log 2 \times x} + \frac{1}{2}\log(2\pi/\log 2/x) + \frac{1}{24}\log 2 \times x$$
$$+S\left(\frac{4\pi^2}{x(\log 2)^2}\right).$$

The fixed point at $x = 2\pi/\log 2$ of the recursion does not produce a value here, but we get by way of confirming the computation that

$$0 = -\frac{1}{6}\frac{\pi^2}{2\pi} + \frac{1}{24}2\pi.$$

This was math.stackexchange.com problem 491948.

3 Some sum formulae by Apery and Plouffe

Suppose we are trying to prove that

$$\begin{aligned}
\frac{3}{2}\zeta(3) &= \frac{\pi^3}{24}\sqrt{2} - 2\sum_{k=1}^{\infty}\frac{1}{k^3(e^{\pi k\sqrt{2}}-1)} - \sum_{k=1}^{\infty}\frac{1}{k^3(e^{2\pi k\sqrt{2}}-1)}\\
\frac{3}{2}\zeta(5) &= \frac{\pi^5}{270}\sqrt{2} - 4\sum_{k=1}^{\infty}\frac{1}{k^5(e^{\pi k\sqrt{2}}-1)} + \sum_{k=1}^{\infty}\frac{1}{k^5(e^{2\pi k\sqrt{2}}-1)}\\
\frac{9}{2}\zeta(7) &= \frac{41\pi^7}{37800}\sqrt{2} - 8\sum_{k=1}^{\infty}\frac{1}{k^7(e^{\pi k\sqrt{2}}-1)} - \sum_{k=1}^{\infty}\frac{1}{k^7(e^{2\pi k\sqrt{2}}-1)}
\end{aligned}$$

Introduce the sum

$$S(x;\alpha,p) = \sum_{n\geq 1}\frac{1}{n^p(e^{\alpha n x}-1)}$$

with p a positive odd integer and $\alpha > 1$, so that we seek e.g. $2S(1;\pi\sqrt{2},3) + S(1;2\pi\sqrt{2},3)$.

The sum term is harmonic and may be evaluated by inverting its Mellin transform.

Recall the harmonic sum identity (this is the last time it will be quoted in the present document)

$$\mathfrak{M}\left(\sum_{k\geq 1}\lambda_k g(\mu_k x); s\right) = \left(\sum_{k\geq 1}\frac{\lambda_k}{\mu_k^s}\right) g^*(s)$$

where $g^*(s)$ is the Mellin transform of $g(x)$.

In the present case we have

$$\lambda_k = \frac{1}{k^p}, \quad \mu_k = k \quad \text{and} \quad g(x) = \frac{1}{e^{\alpha x}-1}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is

$$\begin{aligned}
\int_0^\infty \frac{1}{e^{\alpha x}-1}x^{s-1}dx &= \int_0^\infty \frac{e^{-\alpha x}}{1-e^{-\alpha x}}x^{s-1}dx\\
&= \int_0^\infty \sum_{q\geq 1} e^{-\alpha q x}x^{s-1}dx = \sum_{q\geq 1}\int_0^\infty e^{-\alpha q x}x^{s-1}dx\\
&= \Gamma(s)\sum_{q\geq 1}\frac{1}{(\alpha q)^s} = \frac{1}{\alpha^s}\Gamma(s)\zeta(s).
\end{aligned}$$

It follows that the Mellin transform $Q(s)$ of the harmonic sum $S(x;\alpha,p)$ is given by

$$Q(s) = \frac{1}{\alpha^s}\Gamma(s)\zeta(s)\zeta(s+p) \quad \text{because} \quad \sum_{k\geq 1} \frac{\lambda_k}{\mu_k^s} = \sum_{k\geq 1} \frac{1}{k^p}\frac{1}{k^s} = \zeta(s+p)$$

for $\Re(s) > 1-p$.

The Mellin inversion integral here is

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Q(s)/x^s ds$$

which we evaluate by shifting it to the left for an expansion about zero.

3.1 First formula

We take

$$Q(s) = \frac{1}{\pi^s \sqrt{2}^s}\left(2 + \frac{1}{2^s}\right)\Gamma(s)\zeta(s)\zeta(s+3).$$

We shift the Mellin inversion integral to the line $s = -1$, integrating right through the pole at $s = -1$ picking up the following residues:

$$\mathrm{Res}(Q(s)/x^s; s = 1) = \frac{\pi^3\sqrt{2}}{72x} \quad \text{and} \quad \mathrm{Res}(Q(s)/x^s; s = 0) = -\frac{3}{2}\zeta(3)$$

and

$$\frac{1}{2}\mathrm{Res}(Q(s)/x^s; s = -1) = \frac{\pi^3\sqrt{2}x}{36}.$$

This almost concludes the proof of the first formula if we can show that the integral on the line $\Re(s) = -1$ vanishes when $x = 1$. To accomplish this we must show that the integrand is odd on this line.

Put $s = -1 - it$ in the integrand to get

$$\pi^{1+it}\sqrt{2}^{1+it}\left(2 + 2^{1+it}\right)\Gamma(-1-it)\zeta(-1-it)\zeta(2-it).$$

Now use the functional equation of the Riemann Zeta function in the following form:

$$\zeta(1-s) = \frac{2}{2^s\pi^s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

to obtain (with $s = -1 - it$)

$$\pi^{1+it}\sqrt{2}^{1+it}\left(2 + 2^{1+it}\right)\zeta(2+it)2^{-1-it}\pi^{-1-it}\frac{1}{2\cos\left(\frac{\pi(-1-it)}{2}\right)}\zeta(2-it)$$

which is

$$\sqrt{2}^{1+it}\left(2^{-it} + 1\right)\zeta(2+it)\frac{1}{2\cos\left(\frac{\pi(1+it)}{2}\right)}\zeta(2-it)$$

and finally yields

$$-\frac{1}{\sin(\pi it/2)}\left(\sqrt{2}^{-1-it}+\sqrt{2}^{-1+it}\right)\zeta(2+it)\zeta(2-it).$$

It is now possible to conclude by inspection: the zeta function terms and the powers of the square root are even in t and the sine term is odd, so the whole term is odd and the integral vanishes. (We get exponential decay from the sine term.)

3.2 Second formula

We take

$$Q(s)=\frac{1}{\pi^s\sqrt{2}^s}\left(4-\frac{1}{2^s}\right)\Gamma(s)\zeta(s)\zeta(s+5).$$

We shift the Mellin inversion integral to the line $s=-2$ (no pole on the line this time) picking up the following residues:

$$\mathrm{Res}(Q(s)/x^s; s=1)=\frac{\pi^5\sqrt{2}}{540x} \quad \text{and} \quad \mathrm{Res}(Q(s)/x^s; s=0)=-\frac{3}{2}\zeta(5)$$

and

$$\mathrm{Res}(Q(s)/x^s; s=-1)=\frac{\pi^5\sqrt{2}x}{540}.$$

It remains to verify that the integrand on the line $\Re(s)=-2$ is odd when $x=1$. Put $s=-2-it$ in the integrand to get

$$\pi^{2+it}\sqrt{2}^{2+it}\left(4-2^{2+it}\right)\Gamma(-2-it)\zeta(-2-it)\zeta(3-it).$$

Applying the functional equation once again with $s=-2-it$ we obtain

$$\pi^{2+it}\sqrt{2}^{2+it}\left(4-2^{2+it}\right)\zeta(3+it)2^{-2-it}\pi^{-2-it}\frac{1}{2\cos\left(\frac{\pi(-2-it)}{2}\right)}\zeta(3-it)$$

which is

$$\sqrt{2}^{2+it}\left(2^{-it}-1\right)\zeta(3+it)\frac{1}{2\cos\left(\frac{\pi(-2-it)}{2}\right)}\zeta(3-it)$$

which is in turn

$$\left(\sqrt{2}^{2-it}-\sqrt{2}^{2+it}\right)\frac{1}{2\cos\left(\frac{\pi(2+it)}{2}\right)}\zeta(3+it)\zeta(3-it)$$

which finally yields

$$-\left(\sqrt{2}^{2-it}-\sqrt{2}^{2+it}\right)\frac{1}{2\cos(\pi it/2)}\zeta(3+it)\zeta(3-it)$$

The product of the zeta function terms is even, as is the cosine term. The term in front is odd, so the integrand is odd as claimed. (As before we get exponential decay from the cosine term.)

3.3 Third formula

We take

$$Q(s) = \frac{1}{\pi^s \sqrt{2}^s} \left(8 + \frac{1}{2^s} \right) \Gamma(s)\zeta(s)\zeta(s+7).$$

We shift the Mellin inversion integral to the line $s = -3$, integrating right through the pole at $s = -3$ picking up the following residues:

$$\mathrm{Res}(Q(s)/x^s; s = 1) = \frac{17\pi^7\sqrt{2}}{37800x} \quad \text{and} \quad \mathrm{Res}(Q(s)/x^s; s = 0) = -\frac{9}{2}\zeta(7)$$

and

$$\mathrm{Res}(Q(s)/x^s; s = -1) = \frac{\pi^7\sqrt{2}x}{1134} \quad \text{and} \quad \frac{1}{2}\mathrm{Res}(Q(s)/x^s; s = -3) = -\frac{\pi^7\sqrt{2}x^3}{4050}.$$

This almost concludes the proof of this third formula if we can show that the integral on the line $\Re(s) = -3$ vanishes when $x = 1$. To accomplish this we must show once more that the integrand is odd on this line.

Put $s = -3 - it$ in the integrand to get

$$\pi^{3+it}\sqrt{2}^{3+it} \left(8 + 2^{3+it}\right) \Gamma(-3-it)\zeta(-3-it)\zeta(4-it).$$

By the functional equation we obtain with $s = -3 - it$

$$\pi^{3+it}\sqrt{2}^{3+it} \left(8 + 2^{3+it}\right) \zeta(4+it) 2^{-3-it}\pi^{-3-it} \frac{1}{2\cos\left(\pi(-3-it)/2\right)} \zeta(4-it)$$

which is

$$\sqrt{2}^{3+it} \left(2^{-it} + 1\right) \zeta(4+it) \frac{1}{2\cos\left(\pi(3+it)/2\right)} \zeta(4-it)$$

which finally yields

$$\frac{1}{2\sin(\pi it/2)} \left(\sqrt{2}^{3-it} + \sqrt{2}^{3+it} \right) \zeta(4+it)\zeta(4-it).$$

This concludes it since the two zeta function terms together are even as is the square root term while the sine term is odd, so their product is odd.

This is math.stackexchange.com question 157040.

4 A sum formula by Hardy and Ramanujan

Suppose we are trying to evaluate

$$\sum_{n\ge 1} \left(\coth(n\pi x) + x^2 \coth(n\pi/x)\right)/n^3.$$

Put

$$S(x) = \zeta(3) + \sum_{n\geq 1} \frac{-1+\coth(n\pi x)}{n^3}$$

and introduce the sum

$$T(x) = \sum_{n\geq 1} \frac{-1+\coth(n\pi x)}{n^3}.$$

The sum term is harmonic and may be evaluated by inverting its Mellin transform. We will construct a functional equation for $T(x)$.

Using the harmonic sum identity in the present case we have

$$\lambda_k = \frac{1}{k^3}, \quad \mu_k = k \quad \text{and} \quad g(x) = 2\frac{e^{-2\pi x}}{1-e^{-2\pi x}}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is

$$\begin{aligned} 2\int_0^\infty & \frac{e^{-2\pi x}}{1-e^{-2\pi x}} x^{s-1} dx \\ &= 2\int_0^\infty \sum_{q\geq 1} e^{-2q\pi x} x^{s-1} dx = 2\sum_{q\geq 1}\int_0^\infty e^{-2q\pi x} x^{s-1} dx \\ &= 2\Gamma(s)\sum_{q\geq 1}\frac{1}{(2\pi q)^s} = \frac{2}{2^s}\frac{1}{\pi^s}\Gamma(s)\zeta(s). \end{aligned}$$

It follows that the Mellin transform $Q(s)$ of the harmonic sum $T(x)$ is given by

$$Q(s) = \frac{2}{2^s}\frac{1}{\pi^s}\Gamma(s)\zeta(s)\zeta(s+3) \quad \text{because} \quad \sum_{k\geq 1}\frac{\lambda_k}{\mu_k^s} = \sum_{k\geq 1}\frac{1}{k^3}\frac{1}{k^s} = \zeta(s+3)$$

for $\Re(s) > -2$.

The Mellin inversion integral here is

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Q(s)/x^s ds$$

which we evaluate by shifting it to the left for an expansion about zero.

Fortunately the trivial zeros of the two zeta function terms cancel the poles of the gamma function term. Shifting to $\Re(s) = -3 - 1/2$ we get

$$T(x) = \frac{\pi^3 x^3}{90} + 4\zeta'(-2)\pi^2 x^2 + \frac{\pi^3 x}{18} - \zeta(3) + \frac{\pi^3}{90x} + \frac{1}{2\pi i}\int_{-7/2-i\infty}^{-7/2+i\infty} Q(s)/x^s ds.$$

Substitute $s = -2 - t$ in the remainder integral to get

$$-\frac{1}{2\pi i}\int_{3/2+i\infty}^{3/2-i\infty} \frac{2}{2^{-2-t}}\frac{1}{\pi^{-2-t}}\Gamma(-2-t)\zeta(-2-t)\zeta(1-t)x^{t+2} dt$$

which is

$$\frac{x^2}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} 2^{3+t}\pi^{2+t}\Gamma(-2-t)\zeta(-2-t)\zeta(1-t)x^t dt.$$

In view of the desired functional equation we now use the functional equation of the Riemann zeta function on $Q(s)$ to prove that the integrand of the last integral is in fact $-Q(t)$.

Start with the functional equation

$$\zeta(1-s) = \frac{2}{2^s\pi^s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

and substitute this into $Q(s)$ to obtain

$$Q(s) = \frac{2}{2^s}\frac{1}{\pi^s}\frac{\zeta(1-s)2^s\pi^s}{2\cos\left(\frac{\pi s}{2}\right)}\zeta(s+3) = \frac{\zeta(3+s)}{\cos\left(\frac{\pi s}{2}\right)}\zeta(1-s).$$

Apply the functional equation again (this time to $\zeta(s+3)$) to get

$$Q(s) = \frac{1}{\cos\left(\frac{\pi s}{2}\right)}\frac{2}{2^{-2-s}\pi^{-2-s}}\cos\left(\frac{\pi(-2-s)}{2}\right)\Gamma(-2-s)\zeta(-2-s)\zeta(1-s)$$

Observe that

$$\frac{\cos\left(-\pi-\frac{\pi s}{2}\right)}{\cos\left(\frac{\pi s}{2}\right)} = -\frac{\cos\left(-\frac{\pi s}{2}\right)}{\cos\left(\frac{\pi s}{2}\right)} = -1$$

so we finally get

$$Q(s) = -2^{3+s}\pi^{2+s}\Gamma(-2-s)\zeta(-2-s)\zeta(1-s),$$

thus proving the claim.

We have established the functional equation

$$T(x) = \frac{\pi^3 x^3}{90} + 4\zeta'(-2)\pi^2 x^2 + \frac{\pi^3 x}{18} - \zeta(3) + \frac{\pi^3}{90x} - x^2 T(1/x).$$

Finally returning to the sum that was the initial goal we see that it has the value

$$\zeta(3) + T(x) + x^2(\zeta(3) + T(1/x))$$

or

$$\zeta(3) + T(x) + x^2\zeta(3) + x^2 T(1/x).$$

Using the functional equation for $T(x)$ this becomes

$$\zeta(3) + T(x) + x^2\zeta(3) + \frac{\pi^3 x^3}{90} + 4\zeta'(-2)\pi^2 x^2 + \frac{\pi^3 x}{18} - \zeta(3) + \frac{\pi^3}{90x} - T(x)$$

which is

$$x^2\zeta(3) + \frac{\pi^3 x^3}{90} + 4\zeta'(-2)\pi^2 x^2 + \frac{\pi^3 x}{18} + \frac{\pi^3}{90x}.$$

In view of the fact that

$$\zeta(3) + 4\zeta'(-2)\pi^2 = 0$$

this finally becomes

$$\frac{\pi^3 x^3}{90} + \frac{\pi^3 x}{18} + \frac{\pi^3}{90x} = \frac{\pi^3}{90x}\left(x^4 + 5x^2 + 1\right).$$

This is math.stackexchange.com question 907480.

5 A transform with remarkable symmetries

Suppose we seek to show that

$$\sum_{n\ge 1} \frac{(-1)^{n+1}}{n^3} \frac{1}{\sinh(\pi n)} = \frac{\pi^3}{360}.$$

Using

$$\frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} = 2\frac{e^{-x}}{1-e^{-2x}}$$

this is the same as

$$\sum_{n\ge 1} \frac{(-1)^{n+1}}{n^3} \frac{e^{-n\pi}}{1-e^{-2n\pi}} = \frac{\pi^3}{720}.$$

Let p be a positive integer and introduce

$$S(x;p) = \sum_{n\ge 1} \frac{(-1)^{n+1}}{n^{2p+1}} \frac{e^{-nx}}{1-e^{-2nx}}.$$

We will evaluate $S(\pi;p)$ using a functional equation for $S(x;p)$ that is obtained by inverting its Mellin transform.

With the standard harmonic sum identity in the present case we have

$$\lambda_k = \frac{(-1)^{k+1}}{k^{2p+1}}, \quad \mu_k = k \quad \text{and} \quad g(x) = \frac{e^{-x}}{1-e^{-2x}}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is

$$\int_0^\infty \frac{e^{-x}}{1-e^{-2x}} x^{s-1} dx = \int_0^\infty \sum_{q\ge 0} e^{-(2q+1)x} x^{s-1} dx = \sum_{q\ge 0} \int_0^\infty e^{-(2q+1)x} x^{s-1} dx$$

$$= \Gamma(s) \sum_{q\ge 0} \frac{1}{(2q+1)^s} = \left(1 - \frac{1}{2^s}\right)\Gamma(s)\zeta(s)$$

with fundamental strip $\langle 1, \infty\rangle$.

It follows that the Mellin transform $Q(s)$ of the harmonic sum $S(x;p)$ is given by

$$Q(s) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{2^{s+2p}}\right)\Gamma(s)\zeta(s)\zeta(s+2p+1)$$

$$\text{because} \quad \sum_{k\geq 1} \frac{\lambda_k}{\mu_k^s} = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k^{2p+1}} \frac{1}{k^s} = \left(1 - \frac{2}{2^{s+2p+1}}\right)\zeta(s+2p+1)$$

for $\Re(s+2p+1) > 1$ or $\Re(s) > -2p$.

The Mellin inversion integral here is

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Q(s)/x^s ds$$

which we evaluate by shifting it to the left for an expansion about zero.

Fortunately the trivial zeros of the two zeta function terms cancel the poles of the gamma function term. The first term cancels those at $-2m$ where $m \geq 1$ and the second one the odd ones from $-2p-3$ on, which leaves the poles at $s = 1$, and $-2q-1$ with $0 \leq q \leq p$. It would appear there is a pole at $s = -2p$ but this is not the case since we have two simple poles among the five terms but also two zero values, making for cancelation. The pole at $s = 0$ is canceled as well.

For the residue at $s = 1$ we find

$$\frac{1}{2}\frac{2^{2p+1}-1}{2^{2p+1}} \times 1 \times \zeta(2p+2)\frac{1}{x} = \frac{2^{2p+1}-1}{2^{2p+2}} \frac{(-1)^p B_{2p+2}(2\pi)^{2p+2}}{2(2p+2)!}\frac{1}{x}$$

$$= (2^{2p+1}-1)\frac{(-1)^p B_{2p+2}\pi^{2p+2}}{2(2p+2)!}\frac{1}{x}.$$

The negative odd values at $s = -2q-1$ yield

$$\left(1 - \frac{1}{2^{-2q-1}}\right)\left(1 - \frac{1}{2^{2p-2q-1}}\right)\frac{(-1)^{2q+1}}{(2q+1)!}\zeta(-2q-1)\zeta(2p-2q)x^{2q+1}$$

$$= (1-2^{2q+1})\left(1 - \frac{1}{2^{2p-2q-1}}\right)\frac{1}{(2q+1)!}\frac{B_{2q+2}}{2q+2}\frac{(-1)^{p-q+1}B_{2p-2q}(2\pi)^{2p-2q}}{2(2p-2q)!}x^{2q+1}$$

$$= \frac{1}{2}(1-2^{2q+1})(2^{2p-2q-1}-1)\frac{(-1)^{p-q+1}}{(2q+1)!}\frac{B_{2q+2}B_{2p-2q}\pi^{2p-2q}}{(2p-2q)!(q+1)}x^{2q+1}.$$

Shifting to $\Re(s) = -2p-3/2$ we get

$$S(x;p) = (2^{2p+1}-1)\frac{(-1)^p B_{2p+2}\pi^{2p+2}}{2(2p+2)!}\frac{1}{x}$$

$$+\frac{1}{2}\sum_{q=0}^{p}(1-2^{2q+1})(2^{2p-2q-1}-1)\frac{(-1)^{p-q+1}}{(2q+1)!}\frac{B_{2q+2}B_{2p-2q}\pi^{2p-2q}}{(2p-2q)!(q+1)}x^{2q+1}$$

$$+\frac{1}{2\pi i}\int_{-2p-3/2-i\infty}^{-2p-3/2+i\infty} Q(s)/x^s ds.$$

We will turn this into the promised functional equation.

Substitute $s=-2p-t$ in the remainder integral to get

$$-\frac{1}{2\pi i}\int_{3/2+i\infty}^{3/2-i\infty}\left(1-2^{2p+t}\right)\left(1-2^t\right)\Gamma(-2p-t)\zeta(-2p-t)\zeta(1-t)x^{t+2p}dt$$

which is

$$\frac{x^{2p}}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\left(1-2^{2p+t}\right)\left(1-2^t\right)\Gamma(-2p-t)\zeta(-2p-t)\zeta(1-t)x^{t}dt.$$

In view of the desired functional equation we now use the functional equation of the Riemann zeta function on $Q(s)$ to prove that the integrand of the last integral is in fact $(-1)^p Q(t)/\pi^{2p+2t}$.

Start with the functional equation

$$\zeta(1-s)=\frac{2}{2^s\pi^s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

and substitute this into $Q(s)$ to obtain

$$\begin{aligned}Q(s)&=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{2^{s+2p}}\right)\frac{\zeta(1-s)2^s\pi^s}{2\cos\left(\frac{\pi s}{2}\right)}\zeta(s+2p+1)\\&=\frac{1}{2}(2^s-1)\left(1-\frac{1}{2^{s+2p}}\right)\pi^s\frac{\zeta(s+2p+1)}{\cos\left(\frac{\pi s}{2}\right)}\zeta(1-s).\end{aligned}$$

Apply the functional equation again (this time to $\zeta(s+2p+1)$) to get

$$\begin{aligned}Q(s)&=\frac{1}{2}\frac{\pi^s}{\cos\left(\frac{\pi s}{2}\right)}(2^s-1)\left(1-\frac{1}{2^{s+2p}}\right)\frac{2}{2^{-2p-s}\pi^{-2p-s}}\cos\left(\frac{\pi(-2p-s)}{2}\right)\\&\quad\times\Gamma(-2p-s)\zeta(-2p-s)\zeta(1-s)\\&=\frac{\pi^s}{\cos\left(\frac{\pi s}{2}\right)}(2^s-1)(2^{2p+s}-1)\pi^{2p+s}(-1)^p\cos\left(\frac{-\pi s}{2}\right)\\&\quad\times\Gamma(-2p-s)\zeta(-2p-s)\zeta(1-s)\end{aligned}$$

and we finally get

$$Q(s)=(-1)^p\pi^{2p+2s}(1-2^s)(1-2^{2p+s})\Gamma(-2p-s)\zeta(-2p-s)\zeta(1-s)$$

thus proving the claim.

Return to the remainder integral and re-write it as follows:

$$(-1)^p \frac{(x/\pi)^{2p}}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} (-1)^p \pi^{2p+2t} \left(1-2^{2p+t}\right)\left(1-2^t\right)$$
$$\times\Gamma(-2p-t)\zeta(-2p-t)\zeta(1-t)(x/\pi^2)^t dt.$$

so that the fact of it being a multiple of the defining integral of $S(\pi^2/x;p)$ becomes readily apparent.

We have established the functional equation

$$S(x;p) = (2^{2p+1}-1)\frac{(-1)^p B_{2p+2}\pi^{2p+2}}{2(2p+2)!}\frac{1}{x}$$
$$+\frac{1}{2}\sum_{q=0}^{p}(1-2^{2q+1})(2^{2p-2q-1}-1)\frac{(-1)^{p-q+1}}{(2q+1)!}\frac{B_{2q+2}B_{2p-2q}\pi^{2p-2q}}{(2p-2q)!(q+1)}x^{2q+1}$$
$$+(-1)^p\left(\frac{x}{\pi}\right)^{2p} S(\pi^2/x;p).$$

Now the value $x = \pi$ is obviously special here (fixed point) and we get for $p = 2r+1$ with $r \ge 0$ (p even yields a Bernoulli number identity)

$$S(\pi;2r+1) = \sum_{n\ge 1}\frac{(-1)^{n+1}}{n^{4r+3}}\frac{e^{-nx}}{1-e^{-2nx}} = -\frac{\pi^{4r+3}}{4}(2^{4r+3}-1)\frac{B_{4r+4}}{(4r+4)!}$$
$$+\frac{\pi^{4r+3}}{4}\sum_{q=0}^{2r+1}(1-2^{2q+1})(2^{4r+1-2q}-1)\frac{(-1)^q}{(2q+1)!}\frac{B_{2q+2}B_{4r+2-2q}}{(4r+2-2q)!(q+1)}.$$

We obtain a rational multiple of π^{4r+3}. Scale by two to get for

$$\sum_{n\ge 1}\frac{(-1)^{n+1}}{n^3}\frac{1}{\sinh(\pi n)}, \quad \sum_{n\ge 1}\frac{(-1)^{n+1}}{n^7}\frac{1}{\sinh(\pi n)},$$
$$\sum_{n\ge 1}\frac{(-1)^{n+1}}{n^{11}}\frac{1}{\sinh(\pi n)}, \quad \sum_{n\ge 1}\frac{(-1)^{n+1}}{n^{15}}\frac{1}{\sinh(\pi n)}, \quad \ldots$$

the values

$$\frac{\pi^3}{360}, \quad \frac{13\,\pi^7}{453600}, \quad \frac{4009\,\pi^{11}}{13621608000}, \quad \frac{13739\,\pi^{15}}{4547140416000}, \quad \ldots$$

These are dominated by the first term

$$\frac{1}{\sinh(\pi)}.$$

This was math.stackexchange.com problem 2598443.

6 A sum formula by Cauchy and Ramanujan

We are trying to show that

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\coth m\pi}{m^{4p+3}} = (2\pi)^{4p+3} \sum_{q=0}^{2p+2} \frac{B_{2q}}{(2q)!}(-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!},$$

Note that this sum is in fact

$$2\sum_{m\geq 1} \frac{\coth m\pi}{m^{4p+3}}.$$

The sum term

$$T_p(x) = 2\sum_{m\geq 1} \frac{\coth mx}{m^{4p+3}}$$

is harmonic and may be evaluated by inverting its Mellin transform.

Observe that

$$\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = 1 + 2\frac{e^{-x}}{e^x-e^{-x}} = 1 + 2\frac{e^{-2x}}{1-e^{-2x}}.$$

This yields

$$T_p(x) = 2\zeta(4p+3) + 4\sum_{m\geq 1} \frac{1}{m^{4p+3}} \frac{e^{-2mx}}{1-e^{-2mx}}.$$

We will now work with

$$S(x) = \sum_{m\geq 1} \frac{1}{m^{4p+3}} \frac{e^{-2mx}}{1-e^{-2mx}}$$

and establish a functional equation for $S(x)$ that has $x = \pi$ as a fixed point.

With the harmonic sum identity in the present case we have

$$\lambda_k = \frac{1}{k^{4p+3}}, \quad \mu_k = k \quad \text{and} \quad g(x) = \frac{\exp(-2x)}{1-\exp(-2x)}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which we compute as follows:

$$\int_0^\infty \frac{\exp(-2x)}{1-\exp(-2x)} x^{s-1} dx = \int_0^\infty \sum_{q\geq 1} \exp(-2qx)\, x^{s-1} dx$$
$$= \sum_{q\geq 1} \int_0^\infty \exp(-2qx)\, x^{s-1} dx = \Gamma(s) \sum_{q\geq 1} \frac{1}{2^s q^s} = \frac{1}{2^s}\Gamma(s)\zeta(s).$$

with fundamental strip $\langle 1, \infty\rangle$.

Hence the Mellin transform $Q(s)$ of $S(x)$ is given by

$$Q(s) = \frac{1}{2^s}\Gamma(s)\zeta(s)\zeta(s+4p+3) \quad \text{because} \quad \sum_{k\geq 1} \frac{\lambda_k}{\mu_k^s} = \zeta(s+4p+3)$$

where $\Re(s+4p+3) > 1$ or $\Re(s) > -4p-2$.

Intersecting the fundamental strip and the half-plane from the zeta function term we find that the Mellin inversion integral for an expansion about zero is

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Q(s)/x^s ds$$

which we evaluate in the left half-plane $\Re(s) < 3/2$.

The plain zeta function term cancels the poles of the gamma function term at even negative integers $2q \leq -2$ and the compound zeta function term the poles at odd negative integers $2q+1 \leq -4p-5$. We are left with the contributions from $s=1$, $s=0$ and $s=-1$ which are

$$\begin{aligned} \mathrm{Res}(Q(s)/x^s; s=1) &= \frac{1}{2x}\zeta(4p+4) \\ \mathrm{Res}(Q(s)/x^s; s=0) &= -\frac{1}{2}\zeta(4p+3) \\ \mathrm{Res}(Q(s)/x^s; s=-1) &= \frac{1}{6}x\zeta(4p+2). \end{aligned}$$

The remaining contributions are

$$\sum_{q=1}^{2p+1} \mathrm{Res}(Q(s)/x^s; s=-2q-1)$$

$$= \sum_{q=1}^{2p+1} 2^{2q+1}x^{2q+1}\frac{(-1)^{2q+1}}{(2q+1)!}\zeta(-2q-1)\zeta(4p+2-2q)$$

$$= \sum_{q=1}^{2p+1} 2^{2q+1}x^{2q+1}\frac{1}{(2q+1)!}\frac{B_{2q+2}}{2q+2}(-1)^{2p+1-q+1}\frac{B_{4p+2-2q}(2\pi)^{4p+2-2q}}{2(4p+2-2q)!}$$

$$= \sum_{q=1}^{2p+1} 2^{2q+1}x^{2q+1}\frac{B_{2q+2}}{(2q+2)!}(-1)^{-q}\frac{B_{4p+2-2q}(2\pi)^{4p+2-2q}}{2(4p+2-2q)!}$$

$$= \sum_{q=2}^{2p+2} 2^{2q-1}x^{2q-1}\frac{B_{2q}}{(2q)!}(-1)^{q+1}\frac{B_{4p+4-2q}(2\pi)^{4p+4-2q}}{2(4p+4-2q)!}$$

$$= 2^{4p+2}\sum_{q=2}^{2p+2} x^{2q-1}\frac{B_{2q}}{(2q)!}(-1)^{q+1}\frac{B_{4p+4-2q}\pi^{4p+4-2q}}{(4p+4-2q)!}$$

$$= 2^{4p+2}\sum_{q=2}^{2p+2} x^{2q-1}\pi^{4p+4-2q}\frac{B_{2q}}{(2q)!}(-1)^{q+1}\frac{B_{4p+4-2q}}{(4p+4-2q)!}$$

and the one from the pole of the compound zeta function term at $s = -4p-2$ which we'll do in a moment.

Now some algebra shows that setting $q = 0$ and $q = 1$ in the sum produces precisely the values that we obtained earlier for the poles at $s = 1$ and $s = -1$ so we may extend the sum to start at zero, keeping only the residue from the pole at $s = 0$ to get

$$2^{4p+2} \sum_{q=0}^{2p+2} x^{2q-1} \pi^{4p+4-2q} \frac{B_{2q}}{(2q)!} (-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!}.$$

We have thus established that

$$\begin{aligned} S(x) = -\frac{1}{2}\zeta(4p+3) + \mathrm{Res}(Q(s)/x^s; s = -4p-2) \\ + 2^{4p+2} \sum_{q=0}^{2p+2} x^{2q-1} \pi^{4p+4-2q} \frac{B_{2q}}{(2q)!} (-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!} \\ + \frac{1}{2\pi i} \int_{-4p-4-i\infty}^{-4p-4+i\infty} Q(s)/x^s ds. \end{aligned}$$

To treat the integral recall the duplication formula of the gamma function:

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right).$$

which yields for the integral

$$\int_{-4p-4-i\infty}^{-4p-4+i\infty} \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \zeta(s)\zeta(s+4p+3)/x^s ds.$$

Furthermore observe the following variant of the functional equation of the Riemann zeta function:

$$\Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

which gives for the integral

$$\begin{aligned} \int_{-4p-4-i\infty}^{-4p-4+i\infty} \frac{\pi^{s-1}}{2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)\zeta(s+4p+3)/x^s ds \\ = \int_{-4p-4-i\infty}^{-4p-4+i\infty} \frac{\pi^s}{2} \frac{1}{\sin(\pi/2(s+1))} \zeta(1-s)\zeta(s+4p+3)/x^s ds. \end{aligned}$$

Now put $s = -4p - 2 - u$ to get

$$\int_{2-i\infty}^{2+i\infty} \frac{\pi^{-4p-2-u}}{2} \frac{1}{\sin(\pi/2(-4p-2-u+1))} \times \zeta(u+4p+3)\zeta(1-u)/x^{-4p-2-u} du$$
$$= \frac{x^{4p+2}}{\pi^{4p+2}} \int_{2-i\infty}^{2+i\infty} \frac{\pi^{-u}}{2} \frac{1}{\sin(\pi/2(-(u+1)))} \times \zeta(u+4p+3)\zeta(1-u)/x^{-u} du$$
$$= -\frac{x^{4p+2}}{\pi^{4p+2}} \int_{2-i\infty}^{2+i\infty} \frac{\pi^{-u}}{2} \frac{1}{\sin(\pi/2(u+1))} \times \zeta(u+4p+3)\zeta(1-u)/x^{-u} du.$$

We have established the functional equation

$$S(x) = -\frac{1}{2}\zeta(4p+3) + \mathrm{Res}(Q(s)/x^s; s = -4p-2)$$
$$+ 2^{4p+2} \sum_{q=0}^{2p+2} x^{2q-1} \pi^{4p+4-2q} \frac{B_{2q}}{(2q)!} (-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!} - \left(\frac{x}{\pi}\right)^{4p+2} S\left(\frac{\pi^2}{x}\right).$$

Setting $x = \pi$ we have

$$S(\pi) = -\frac{1}{2}\zeta(4p+3) + \mathrm{Res}(Q(s)/\pi^s; s = -4p-2)$$
$$+ 2^{4p+2}\pi^{4p+3} \sum_{q=0}^{2p+2} \frac{B_{2q}}{(2q)!} (-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!} - \left(\frac{\pi}{\pi}\right)^{4p+2} S(\pi)$$

and hence

$$S(\pi) = -\frac{1}{4}\zeta(4p+3) + \frac{1}{2}\mathrm{Res}(Q(s)/\pi^s; s = -4p-2)$$
$$+ 2^{4p+1}\pi^{4p+3} \sum_{q=0}^{2p+2} \frac{B_{2q}}{(2q)!} (-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!}.$$

To conclude we treat the residue that we have defered until now. Recall the alternate form of $Q(s)/\pi^s$,

$$\frac{\pi^s}{2} \frac{1}{\sin(\pi/2(s+1))} \zeta(1-s)\zeta(s+4p+3)/\pi^s$$
$$= \frac{1}{2} \frac{1}{\sin(\pi/2(s+1))} \zeta(1-s)\zeta(s+4p+3).$$

It follows that the residue at $s = -4p - 2$ is

$$\frac{1}{2} \frac{1}{\sin(\pi/2(-4p-1))} \zeta(4p+3) = \frac{1}{2} \frac{1}{\sin(-\pi/2)} \zeta(4p+3) = -\frac{1}{2}\zeta(4p+3).$$

This finally yields for $S(\pi)$

$$S(\pi) = -\frac{1}{2}\zeta(4p+3) + 2^{4p+1}\pi^{4p+3} \sum_{q=0}^{2p+2} \frac{B_{2q}}{(2q)!}(-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!}.$$

Computing $T_p(\pi)$ we thus obtain

$$2\zeta(4p+3) - 4 \times \frac{1}{2}\zeta(4p+3) + 2^{4p+3}\pi^{4p+3} \sum_{q=0}^{2p+2} \frac{B_{2q}}{(2q)!}(-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!}$$

or

$$(2\pi)^{4p+3} \sum_{q=0}^{2p+2} \frac{B_{2q}}{(2q)!}(-1)^{q+1} \frac{B_{4p+4-2q}}{(4p+4-2q)!},$$

QED.

This is math.stackexchange.com question 1293232.

7 A functional equation relating two harmonic sums

Suppose we seek a functional equation for

$$S(x) = \sum_{k\ge 1} \frac{1}{(2k-1)} \frac{1}{\exp(x(2k-1)) - \exp(-x(2k-1))}.$$

The sum $S(x)$ is harmonic and may be evaluated by inverting its Mellin transform.

With the harmonic sum identity in the present case we have

$$\lambda_k = \frac{1}{(2k-1)}, \quad \mu_k = 2k-1 \quad \text{and} \quad g(x) = \frac{1}{\exp(x) - \exp(-x)}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is computed as follows:

$$g^*(s) = \int_0^\infty \frac{1}{\exp(x) - \exp(-x)} x^{s-1} dx = \int_0^\infty \frac{\exp(-x)}{1 - \exp(-2x)} x^{s-1} dx$$

$$= \int_0^\infty \sum_{q\ge 0} \exp(-(2q+1)x) x^{s-1} dx = \sum_{q\ge 0} \frac{1}{(2q+1)^s} \Gamma(s) = \left(1 - \frac{1}{2^s}\right) \Gamma(s)\zeta(s).$$

Hence the Mellin transform $Q(s)$ of $S(x)$ is given by

$$Q(s) = \left(1 - \frac{1}{2^{s+1}}\right) \left(1 - \frac{1}{2^s}\right) \Gamma(s)\zeta(s)\zeta(s+1)$$

$$\text{because} \quad \sum_{k\ge 1} \frac{\lambda_k}{\mu_k^s} = \left(1 - \frac{1}{2^{s+1}}\right) \zeta(s+1)$$

where $\Re(s) > 1$. Intersecting the fundamental strip and the half-plane from the zeta function term we find that the Mellin inversion integral for an expansion about zero is

$$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Q(s)/x^s ds$$

which we evaluate in the left half-plane $\Re(s) < 3/2$.

The two zeta function terms cancel the poles of the gamma function term and we are left with just

$$\mathrm{Res}(Q(s)/x^s; s=1) = \frac{\pi^2}{16x} \quad \text{and} \tag{1}$$

$$\mathrm{Res}(Q(s)/x^s; s=0) = -\frac{1}{4}\log 2. \tag{2}$$

This shows that

$$S(x) = \frac{\pi^2}{16x} - \frac{1}{4}\log 2 + \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} Q(s)/x^s ds.$$

To treat the integral recall the duplication formula of the gamma function:

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right).$$

which yields for $Q(s)$

$$\left(1-\frac{1}{2^{s+1}}\right)\left(1-\frac{1}{2^s}\right)\frac{1}{\sqrt{\pi}}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s)\zeta(s+1)$$

Furthermore observe the following variant of the functional equation of the Riemann zeta function:

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

which gives for $Q(s)$

$$\left(1-\frac{1}{2^{s+1}}\right)\left(1-\frac{1}{2^s}\right)\frac{1}{\sqrt{\pi}}2^{s-1}\pi^{s-1/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\zeta(s+1)$$

$$= \left(1-\frac{1}{2^{s+1}}\right)\left(1-\frac{1}{2^s}\right)\frac{1}{\sqrt{\pi}}2^{s-1}\pi^{s-1/2}\frac{\pi}{\sin(\pi(s+1)/2)}\zeta(1-s)\zeta(s+1)$$

$$= \left(1-\frac{1}{2^{s+1}}\right)\left(1-\frac{1}{2^s}\right)2^{s-1}\frac{\pi^s}{\sin(\pi(s+1)/2)}\zeta(1-s)\zeta(s+1).$$

Now put $s=-u$ in the remainder integral to get

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left(1-\frac{2^u}{2}\right)(1-2^u)\,2^{-u-1}\frac{\pi^{-u}}{\sin(\pi(-u+1)/2)}\zeta(1+u)\zeta(1-u)x^u du$$

$$= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\left(1-\frac{2^u}{2}\right)(1-2^u)\,2^{-u-1}$$
$$\times\frac{\pi^u}{\sin(\pi(-u+1)/2)}\zeta(1+u)\zeta(1-u)(x/\pi^2)^u du.$$

We may shift this to 3/2 as there is no pole at $u=1$.
Now

$$\sin(\pi(-u+1)/2) = \sin(\pi(-u-1)/2+\pi)$$
$$= -\sin(\pi(-u-1)/2) = \sin(\pi(u+1)/2)$$

and furthermore

$$\left(1-\frac{2^u}{2}\right)(1-2^u)\,2^{-u-1} = \frac{1}{2}\left(1-\frac{2^u}{2}\right)\left(\frac{1}{2^u}-1\right) = 2^{u-2}\left(\frac{1}{2^{u-1}}-1\right)\left(\frac{1}{2^u}-1\right)$$

$$= 2^{u-2}\left(1-\frac{1}{2^{u-1}}\right)\left(1-\frac{1}{2^u}\right)$$

$$= 2^{u-2}\left(1-\frac{1}{2^{u+1}}\right)\left(1-\frac{1}{2^u}\right) - 3\times 2^{u-2}\frac{1}{2^{u+1}}\left(1-\frac{1}{2^u}\right)$$

$$= \frac{1}{2}2^{u-1}\left(1-\frac{1}{2^{u+1}}\right)\left(1-\frac{1}{2^u}\right) - \frac{3}{4}2^{u-1}\left(1-\frac{1}{2^u}\right)\frac{1}{2^u}.$$

We have shown that

$$S(x) = \frac{\pi^2}{16x} - \frac{1}{4}\log 2 + \frac{1}{2}S(\pi^2/x)$$

$$-\frac{3}{4}\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\left(1-\frac{1}{2^u}\right)\Gamma(u)\zeta(u)\zeta(u+1)(x/\pi^2/2)^u du$$

or alternatively

$$S(x) = \frac{\pi^2}{16x} - \frac{1}{4}\log 2 + \frac{1}{2}S(\pi^2/x) - \frac{3}{4}T(2\pi^2/x)$$

where

$$T(x) = \sum_{k\ge 1}\frac{1}{k}\frac{1}{\exp(kx)-\exp(-kx)}$$

with functional equation

$$T(x) = \frac{1}{24}x - \frac{1}{2}\log 2 + \frac{\pi^2}{12x} - T(2\pi^2/x).$$

which finally yields

$$S(x) = \frac{1}{2}S(\pi^2/x) - \frac{1}{32}x + \frac{1}{8}\log 2 + \frac{3}{4}T(x).$$

Using sinh with

$$S(x) = \sum_{k\ge 1} \frac{1}{(2k-1)} \frac{1}{\sinh((2k-1)x)} \quad \text{and} \quad T(x) = \sum_{k\ge 1} \frac{1}{k} \frac{1}{\sinh(kx)}$$

we obtain the functional equation

$$S(x) = \frac{1}{2}S(\pi^2/x) - \frac{1}{16}x + \frac{1}{4}\log 2 + \frac{3}{4}T(x).$$

We also have

$$T(\sqrt{2}\pi) = \frac{\sqrt{2}\pi}{24} - \frac{1}{2}\log 2 + \frac{\pi\sqrt{2}}{24} = \frac{\sqrt{2}\pi}{12} - \frac{1}{2}\log 2.$$

This was math.stackexchange.com problem 1417849.

8 Functional equation of the Hurwitz Zeta function

Suppose we seek to show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \frac{1}{\exp((2k+1)\pi/2) + \exp(-(2k+1)\pi/2)} = \frac{\pi}{16}.$$

The sum term

$$S(x) = \sum_{k\ge 1} \frac{(-1)^{k+1}}{2k-1} \frac{1}{\exp(x(2k-1)) + \exp(-x(2k-1))}$$

is harmonic and may be evaluated by inverting its Mellin transform. We are interested in $S(\pi/2)$.

With the harmonic sum identity in the present case we have

$$\lambda_k = \frac{(-1)^{k+1}}{2k-1}, \quad \mu_k = 2k-1 \quad \text{and} \quad g(x) = \frac{1}{\exp(x) + \exp(-x)}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is computed as follows:

$$g^*(s) = \int_0^\infty \frac{1}{\exp(x) + \exp(-x)} x^{s-1} dx = \int_0^\infty \frac{\exp(-x)}{1 + \exp(-2x)} x^{s-1} dx$$

$$= \int_0^\infty \sum_{q\ge 0} (-1)^q \exp(-(2q+1)x) x^{s-1} dx = \Gamma(s) \sum_{q\ge 0} \frac{(-1)^q}{(2q+1)^s} = \Gamma(s)\beta(s)$$

where

$$\beta(s) = 4^{-s} \left(\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right) \right).$$

Note that $\beta(s)$ does not have a pole at $s = 1$ and $\beta(1) = \frac{\pi}{4}$.
Hence the Mellin transform $Q(s)$ of $S(x)$ is given by

$$Q(s) = \Gamma(s)\beta(s)\beta(s+1) \quad \text{because} \quad \sum_{k\geq 1} \frac{\lambda_k}{\mu_k^s} = \beta(s+1)$$

where $\Re(s) > 0$. Intersecting the fundamental strip and the half-plane from the zeta function term we find that the Mellin inversion integral for an expansion about zero is

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} Q(s)/x^s ds$$

which we evaluate in the left half-plane $\Re(s) < 1/2$.

The two beta function terms cancel the poles of the gamma function term and we are left with just

$$\mathrm{Res}(Q(s)/x^s; s = 0) = \frac{\pi}{8}.$$

This shows that

$$S(x) = \frac{\pi}{8} + \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} Q(s)/x^s ds.$$

To treat the integral recall the duplication formula of the gamma function:

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right).$$

which yields for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \beta(s)\beta(s+1)$$

Furthermore observe the following variant of the functional equation of the Riemann zeta function adapted to the beta function:

$$\Gamma\left(\frac{s+1}{2}\right) \beta(s) = 2^{1-2s} \pi^{s-1/2} \Gamma\left(1 - \frac{s}{2}\right) \beta(1-s)$$

which gives for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) 2^{1-2s} \pi^{s-1/2} \Gamma\left(1 - \frac{s}{2}\right) \beta(1-s)\beta(s+1)$$

$$= 2^{-s} \pi^{s-1} \frac{\pi}{\sin(\pi s/2)} \beta(1-s)\beta(s+1)$$

$$= 2^{-s}\frac{\pi^s}{\sin(\pi s/2)}\beta(1-s)\beta(s+1).$$

Now put $s = -u$ in the remainder integral to get

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} 2^u \frac{\pi^{-u}}{\sin(-\pi u/2)}\zeta(1+u)\zeta(1-u)x^u du$$

$$= -\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} 2^{-u} \frac{\pi^{u}}{\sin(\pi u/2)}\zeta(1+u)\zeta(1-u)(4x/\pi^2)^u du.$$

We have shown that

$$S(x) = \frac{\pi}{8} - S(\pi^2/4/x).$$

In particular we get

$$S(\pi/2) = \frac{\pi}{8} - S(\pi/2) \quad \text{or} \quad S(\pi/2) = \frac{\pi}{16}$$

as claimed.

Addendum. The functional equation for $\beta(s)$ can be derived from the functional equation of the Hurwitz Zeta function:

$$\zeta\left(1-s,\frac{m}{n}\right) = \frac{2\Gamma(s)}{(2\pi n)^s}\sum_{k=1}^{n}\left[\cos\left(\frac{\pi s}{2}-\frac{2\pi k m}{n}\right)\zeta\left(s,\frac{k}{n}\right)\right].$$

where $1 \le m \le n$.

This yields

$$\zeta(1-s,1/4) = \frac{2\Gamma(s)}{2^{3s}\pi^s}\left[\cos(\pi s/2-\pi/2)\zeta(s,1/4)+\cos(\pi s/2-\pi)\zeta(s,1/2)\right.$$
$$\left.+\cos(\pi s/2-3\pi/2)\zeta(s,3/4)+\cos(\pi s/2-2\pi)\zeta(s)\right]$$
$$= \frac{2\Gamma(s)}{2^{3s}\pi^s}\left[\sin(\pi s/2)\zeta(s,1/4)-\cos(\pi s/2)\zeta(s,1/2)\right.$$
$$\left.-\sin(\pi s/2)\zeta(s,3/4)+\cos(\pi s/2)\zeta(s)\right].$$

Similarly

$$\zeta(1-s,3/4) = \frac{2\Gamma(s)}{2^{3s}\pi^s}\left[\cos(\pi s/2-3\pi/2)\zeta(s,1/4)+\cos(\pi s/2-3\pi)\zeta(s,1/2)\right.$$
$$\left.+\cos(\pi s/2-9\pi/2)\zeta(s,3/4)+\cos(\pi s/2-6\pi)\zeta(s)\right]$$
$$= \frac{2\Gamma(s)}{2^{3s}\pi^s}\left[-\sin(\pi s/2)\zeta(s,1/4)-\cos(\pi s/2)\zeta(s,1/2)\right.$$
$$\left.+\sin(\pi s/2)\zeta(s,3/4)+\cos(\pi s/2)\zeta(s)\right].$$

Subtract to obtain

$$4^{1-s}\beta(1-s) = \frac{2\Gamma(s)}{2^{3s}\pi^s}(2\sin(\pi s/2)\zeta(s,1/4) - 2\sin(\pi s/2)\zeta(s,3/4))$$

or

$$4^{1-s}\beta(1-s) = \frac{2\Gamma(s)}{2^{3s}\pi^s} 2\sin(\pi s/2)4^s\beta(s)$$

which is

$$\beta(1-s) = 2^s\frac{\Gamma(s)}{\pi^s}\sin(\pi s/2)\beta(s)$$

$$= \frac{1}{\pi^{s+1/2}}2^{2s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\sin(\pi s/2)\beta(s)$$

$$= \frac{1}{\pi^{s+1/2}}2^{2s-1}\frac{\pi}{\sin(\pi s/2)}\Gamma\left(1-\frac{s}{2}\right)^{-1}\Gamma\left(\frac{s+1}{2}\right)\sin(\pi s/2)\beta(s)$$

which yields

$$\beta(1-s)\Gamma\left(1-\frac{s}{2}\right) = \pi^{1/2-s}2^{2s-1}\beta(s)\Gamma\left(\frac{s+1}{2}\right)$$

which is the desired result.

This was math.stackexchange.com problem 1447489.

9 Two contrasting examples of fixed point scenarios

Suppose we seek a functional equation for the sum term

$$S(x) = \sum_{k\ge 1}\frac{k^5}{\exp(kx)-1}$$

which is harmonic and may be evaluated by inverting its Mellin transform. We are interested in possible fixed points of the functional equation especially $S(2\pi)$.

With the harmonic sum identity in the present case we have

$$\lambda_k = k^5, \quad \mu_k = k \quad \text{and} \quad g(x) = \frac{1}{\exp(x)-1}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is computed as follows:

$$g^*(s) = \int_0^\infty \frac{1}{\exp(x)-1}x^{s-1}dx = \int_0^\infty \frac{\exp(-x)}{1-\exp(-x)}x^{s-1}dx$$

$$= \int_0^\infty \sum_{q\ge 1}\exp(-qx)x^{s-1}dx = \sum_{q\ge 1}\frac{1}{q^s}\Gamma(s) = \Gamma(s)\zeta(s).$$

Hence the Mellin transform $Q(s)$ of $S(x)$ is given by

$$Q(s) = \Gamma(s)\zeta(s)\zeta(s-5) \quad \text{because} \quad \sum_{k\ge 1} \frac{\lambda_k}{\mu_k^s} = \sum_{k\ge 1} \frac{k^5}{k^s} = \zeta(s-5)$$

where $\Re(s) > 6$.

Intersecting the fundamental strip and the half-plane from the zeta function term we find that the Mellin inversion integral for an expansion about zero is

$$\frac{1}{2\pi i} \int_{13/2-i\infty}^{13/2+i\infty} Q(s)/x^s ds$$

which we evaluate in the left half-plane $\Re(s) < 13/2$.

The two zeta function terms cancel the poles of the gamma function term and we are left with just

$$\text{Res}(Q(s)/x^s; s = 6) = \frac{8\pi^6}{63x^6} \quad \text{and} \text{Res}(Q(s)/x^s; s = 0) \qquad = \frac{1}{504}. \qquad (3)$$

This shows that

$$S(x) = \frac{8\pi^6}{15x^6} + \frac{1}{504} + \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} Q(s)/x^s ds.$$

To treat the integral recall the duplication formula of the gamma function:

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right).$$

which yields for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \zeta(s)\zeta(s-5)$$

Furthermore observe the following variant of the functional equation of the Riemann zeta function:

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

which gives for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1}\pi^{s-1/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\zeta(s-5)$$

$$= \frac{1}{\sqrt{\pi}} 2^{s-1}\pi^{s-1/2} \frac{\pi}{\sin(\pi(s+1)/2)} \zeta(1-s)\zeta(s-5)$$

$$= 2^{s-1} \frac{\pi^s}{\sin(\pi(s+1)/2)} \zeta(1-s)\zeta(s-5).$$

Now put $s = 6 - u$ in the remainder integral to get

$$\frac{1}{x^6}\frac{1}{2\pi i}\int_{13/2-i\infty}^{13/2+i\infty} 2^{5-u}\frac{\pi^{6-u}}{\sin(\pi(7-u)/2)}\zeta(u-5)\zeta(1-u)x^u du$$

$$= \frac{64\pi^6}{x^6}\frac{1}{2\pi i}\int_{13/2-i\infty}^{13/2+i\infty} 2^{u-1}\frac{\pi^u}{\sin(\pi(7-u)/2)}\zeta(u-5)\zeta(1-u)(x/\pi^2/2^2)^u du.$$

Now

$$\sin(\pi(7-u)/2) = \sin(\pi(-u-1)/2 + 4\pi)$$
$$= \sin(\pi(-u-1)/2) = -\sin(\pi(u+1)/2).$$

We have shown that

$$S(x) = \frac{8\pi^6}{63x^6} + \frac{1}{504} - \frac{64\pi^6}{x^6}S(4\pi^2/x).$$

In particular we get

$$S(2\pi) = \frac{1}{63\times 8} + \frac{1}{504} - S(2\pi)$$

or

$$S(2\pi) = \frac{1}{504}.$$

Remark. Unfortunately this method does not work for

$$S(x) = \sum_{k\ge 1}\frac{k^3}{\exp(kx)-1}$$

We get the functional equation

$$S(x) = \frac{\pi^4}{15x^4} - \frac{1}{240} + \frac{16\pi^4}{x^4}S(4\pi^2/x).$$

which yields

$$S(2\pi) = \frac{1}{15\times 16} - \frac{1}{240} + S(2\pi)$$

which holds without providing any data about the value itself.
This was math.stackexchange.com problem 1482918.

10 Making more effective use of a two-cycle of fixed points

Suppose we seek to evaluate

$$F(p) = \sum_{n\ge 1}\frac{n^{4p-1}}{e^{\pi n}-1} - 16^p\sum_{n\ge 1}\frac{n^{4p-1}}{e^{4\pi n}-1}.$$

These sums may be evaluated using harmonic summation techniques.

Introduce the sum

$$S(x;p) = \sum_{n\ge 1} \frac{n^{4p-1}}{e^{nx}-1}$$

with p a positive integer and $x \ge 0$.

The sum term is harmonic and may be evaluated by inverting its Mellin transform.

With the harmonic sum identity in the present case we have

$$\lambda_k = k^{4p-1}, \quad \mu_k = k \quad \text{and} \quad g(x) = \frac{1}{e^x - 1}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is

$$\int_0^\infty \frac{1}{e^x-1} x^{s-1} dx = \int_0^\infty \frac{e^{-x}}{1-e^{-x}} x^{s-1} dx$$
$$= \int_0^\infty \sum_{q\ge 1} e^{-qx} x^{s-1} dx = \sum_{q\ge 1} \int_0^\infty e^{-qx} x^{s-1} dx$$
$$= \Gamma(s) \sum_{q\ge 1} \frac{1}{q^s} = \Gamma(s)\zeta(s).$$

It follows that the Mellin transform $Q(s)$ of the harmonic sum $S(x,p)$ is given by

$$Q(s) = \Gamma(s)\zeta(s)\zeta(s-(4p-1))$$
$$\text{because} \quad \sum_{k\ge 1} \frac{\lambda_k}{\mu_k^s} = \sum_{k\ge 1} k^{4p-1} \frac{1}{k^s} = \zeta(s-(4p-1))$$

for $\Re(s) > 4p$.

The Mellin inversion integral here is

$$\frac{1}{2\pi i} \int_{4p+1/2-i\infty}^{4p+1/2+i\infty} Q(s)/x^s ds$$

which we evaluate by shifting it to the left for an expansion about zero.

The two zeta function terms cancel the poles of the gamma function term and we are left with just

$$\mathrm{Res}(Q(s)/x^s; s=4p) = \Gamma(4p)\zeta(4p) \quad \text{and}$$
$$\mathrm{Res}(Q(s)/x^s; s=0) = \zeta(0)\zeta(-(4p-1)).$$

Computing these residues we get

$$-(4p-1)!\frac{B_{4p}(2\pi)^{4p}}{2(4p)!} = -\frac{B_{4p}(2\pi)^{4p}}{2\times 4p} \quad \text{and} \quad -\frac{1}{2} \times -\frac{B_{4p}}{4p}.$$

This shows that

$$S(x;p) = -\frac{B_{4p}(2\pi)^{4p}}{8p \times x^{4p}} + \frac{B_{4p}}{8p} + \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} Q(s)/x^s ds.$$

To treat the integral recall the duplication formula of the gamma function:

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right).$$

which yields for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s)\zeta(s-(4p-1))$$

Furthermore observe the following variant of the functional equation of the Riemann zeta function:

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

which gives for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1}\pi^{s-1/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\zeta(s-(4p-1))$$

$$= \frac{1}{\sqrt{\pi}} 2^{s-1}\pi^{s-1/2}\frac{\pi}{\sin(\pi(s+1)/2)}\zeta(1-s)\zeta(s-(4p-1))$$

$$= 2^{s-1}\frac{\pi^s}{\sin(\pi(s+1)/2)}\zeta(1-s)\zeta(s-(4p-1)).$$

Now put $s = 4p - u$ in the remainder integral to get

$$\frac{1}{x^{4p}}\frac{1}{2\pi i}\int_{4p+1/2-i\infty}^{4p+1/2+i\infty} 2^{4p-1-u}$$

$$\times\frac{\pi^{4p-u}}{\sin(\pi(4p+1-u)/2)}\zeta(u-(4p-1))\zeta(1-u)x^u du$$

$$= \frac{2^{4p}\pi^{4p}}{x^{4p}}\frac{1}{2\pi i}\int_{4p+1/2-i\infty}^{4p+1/2+i\infty} 2^{u-1}$$

$$\times\frac{\pi^u}{\sin(\pi(4p+1-u)/2)}\zeta(u-(4p-1))\zeta(1-u)(x/\pi^2/2^2)^u du.$$

Now

$$\sin(\pi(4p+1-u)/2) = \sin(\pi(1-u)/2 + 2\pi p)$$

$$= \sin(\pi(1-u)/2) = -\sin(\pi(-1-u)/2) = \sin(\pi(u+1)/2).$$

We have shown that

$$\boxed{S(x;p) = -\frac{B_{4p}(2\pi)^{4p}}{8p \times x^{4p}} + \frac{B_{4p}}{8p} + \frac{2^{4p}\pi^{4p}}{x^{4p}} S(4\pi^2/x;p)}.$$

In particular we get

$$S(\pi;p) = -\frac{B_{4p}2^{4p}}{8p} + \frac{B_{4p}}{8p} + 2^{4p}S(4\pi;p)$$

and

$$S(4\pi;p) = -\frac{B_{4p}2^{-4p}}{8p} + \frac{B_{4p}}{8p} + 2^{-4p}S(\pi;p).$$

Therefore

$$S(\pi;p) - 2^{4p}S(4\pi;p)$$
$$= -\frac{B_{4p}(2^{4p}-1)}{8p} + \frac{B_{4p}}{8p}(1-2^{4p}) + (2^{4p}S(4\pi;p) - S(\pi;p)).$$

We finally conclude that

$$F(p) = \frac{B_{4p}}{4p}(1-2^{4p}) - F(p)$$

or

$$F(p) = \frac{B_{4p}}{8p}(1-2^{4p}).$$

The first few values are

$$\frac{1}{16}, \frac{17}{32}, \frac{691}{16}, \frac{929569}{64}, \frac{221930581}{16}, \ldots$$

We can use the functional equation to extract additional sum formulae. For example when we choose the pair

$$\sqrt{2}\pi \quad \text{and} \quad 2\sqrt{2}\pi$$

the scalar is 2^{2p}. The calculation starts from

$$S(\sqrt{2}\pi;p) = -\frac{B_{4p}2^{2p}}{8p} + \frac{B_{4p}}{8p} + 2^{2p}S(2\sqrt{2}\pi;p)$$

and

$$S(2\sqrt{2}\pi;p) = -\frac{B_{4p}2^{-2p}}{8p} + \frac{B_{4p}}{8p} + 2^{-2p}S(\sqrt{2}\pi;p).$$

Therefore

$$S(\sqrt{2}\pi;p) - 2^{2p} S(2\sqrt{2}\pi;p)$$
$$= -\frac{B_{4p}(2^{2p}-1)}{8p} + \frac{B_{4p}}{8p}(1-2^{2p}) + (2^{2p} S(2\sqrt{2}\pi;p) - S(\sqrt{2}\pi;p)).$$

We obtain the formula

$$\sum_{n\ge 1} \frac{n^{4p-1}}{e^{\sqrt{2}\pi n}-1} - 4^p \sum_{n\ge 1} \frac{n^{4p-1}}{e^{2\sqrt{2}\pi n}-1} = \frac{B_{4p}}{8p}(1-2^{2p}).$$

We get for the initial values (factor is $1+2^{2p}$)

$$\frac{1}{80}, \frac{1}{32}, \frac{691}{1040}, \frac{3617}{64}, \frac{5412941}{400}, \ldots$$

The reader is cordially invited to prove this last result by a different method.

Addendum. To illustrate the creation of identities from the functional equation we present a third example, which is

$$\sqrt{3}\pi \quad \text{and} \quad 4\pi/\sqrt{3}.$$

In this example the scalar is $2^{4p}3^{-2p}$. The calculation starts from

$$S(\sqrt{3}\pi;p) = -\frac{B_{4p}2^{4p}}{8p\times 3^{2p}} + \frac{B_{4p}}{8p} + \frac{2^{4p}}{3^{2p}} S(4\pi/\sqrt{3};p)$$

and

$$S(4\pi/\sqrt{3};p) = -\frac{B_{4p}3^{2p}}{8p\times 2^{4p}} + \frac{B_{4p}}{8p} + \frac{3^{2p}}{2^{4p}} S(\sqrt{3}\pi;p).$$

Therefore

$$S(\sqrt{3}\pi;p) - 2^{4p}3^{-2p} S(4\pi/\sqrt{3};p)$$
$$= -\frac{B_{4p}(2^{4p}3^{-2p}-1)}{8p} + \frac{B_{4p}}{8p}(1-2^{4p}3^{-2p}) + (2^{4p}3^{-2p} S(\sqrt{3}\pi;p) - S(4\pi/\sqrt{3};p)).$$

We obtain the formula

$$\sum_{n\ge 1} \frac{n^{4p-1}}{e^{n\sqrt{3}\pi}-1} - 2^{4p}3^{-2p} \sum_{n\ge 1} \frac{n^{4p-1}}{e^{n4\pi/\sqrt{3}}-1} = \frac{B_{4p}}{8p}(1-2^{4p}3^{-2p}).$$

Remark. The general pattern for

$$\beta \quad \text{and} \quad 4\pi^2/\beta$$

is

$$\boxed{\sum_{n\ge 1} \frac{n^{4p-1}}{e^{n\beta}-1} - \frac{2^{4p}\pi^{4p}}{\beta^{4p}} \sum_{n\ge 1} \frac{n^{4p-1}}{e^{n4\pi^2/\beta}-1} = \frac{B_{4p}}{8p}\left(1 - \frac{2^{4p}\pi^{4p}}{\beta^{4p}}\right).}$$

This was math.stackexchange.com problem 1792052.

11 Double Bernoulli number

Suppose we seek to evaluate

$$S = \sum_{n\geq 1} \frac{n^{13}}{e^{2\pi n} - 1}.$$

This very similar to the previous calculation up to a sign.

Introduce the sum

$$S(x;p) = \sum_{n\geq 1} \frac{n^{4p+1}}{e^{nx} - 1}$$

with p a positive integer and $x > 0$.

Apply the harmonic sum identity to

$$\lambda_k = k^{4p+1}, \quad \mu_k = k \quad \text{and} \quad g(x) = \frac{1}{e^x - 1}.$$

We need the Mellin transform $g^*(s)$ of $g(x)$ which is

$$\int_0^\infty \frac{1}{e^x - 1} x^{s-1} dx = \Gamma(s)\zeta(s).$$

It follows that the Mellin transform $Q(s)$ of the harmonic sum $S(x,p)$ is given by

$$Q(s) = \Gamma(s)\zeta(s)\zeta(s - (4p+1))$$

$$\text{because} \quad \sum_{k\geq 1} \frac{\lambda_k}{\mu_k^s} = \sum_{k\geq 1} k^{4p+1} \frac{1}{k^s} = \zeta(s - (4p+1))$$

for $\Re(s) > 4p+2$.

The Mellin inversion integral here is

$$\frac{1}{2\pi i} \int_{4p+5/2-i\infty}^{4p+5/2+i\infty} Q(s)/x^s ds$$

which we evaluate by shifting it to the left for an expansion about zero.

The two zeta function terms cancel the poles of the gamma function term and we are left with just

$$\mathrm{Res}(Q(s)/x^s; s = 4p+2) = \Gamma(4p+2)\zeta(4p+2)/x^{4p+2} \quad \text{and}$$
$$\mathrm{Res}(Q(s)/x^s; s = 0) = \zeta(0)\zeta(-(4p+1)).$$

Computing these residues we get

$$(4p+1)! \frac{B_{4p+2}(2\pi)^{4p+2}}{2(4p+2)! \times x^{4p+2}} = \frac{B_{4p+2}(2\pi)^{4p+2}}{2 \times (4p+2) \times x^{4p+2}}$$

and

$$-\frac{1}{2} \times -\frac{B_{4p+2}}{4p+2}.$$

This shows that

$$S(x;p) = \frac{B_{4p+2}(2\pi)^{4p+2}}{(8p+4) \times x^{4p+2}} + \frac{B_{4p+2}}{8p+4} + \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} Q(s)/x^s ds.$$

To treat the integral recall the duplication formula of the gamma function:

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right).$$

which yields for $Q(s)$

$$\frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \zeta(s)\zeta(s-(4p+1))$$

Furthermore observe the following variant of the functional equation of the Riemann zeta function:

$$\Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

which gives for $Q(s)$

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}} 2^{s-1} \pi^{s-1/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)\zeta(s-(4p+1)) \\
&= \frac{1}{\sqrt{\pi}} 2^{s-1} \pi^{s-1/2} \frac{\pi}{\sin(\pi(s+1)/2)} \zeta(1-s)\zeta(s-(4p+1)) \\
&= 2^{s-1} \frac{\pi^s}{\sin(\pi(s+1)/2)} \zeta(1-s)\zeta(s-(4p+1)).
\end{aligned}$$

Now put $s = 4p+2-u$ in the remainder integral to get

$$\begin{aligned}
&-\frac{1}{x^{4p+2}} \frac{1}{2\pi i} \int_{4p+5/2+i\infty}^{4p+5/2-i\infty} 2^{4p+1-u} \\
&\times \frac{\pi^{4p+2-u}}{\sin(\pi(4p+3-u)/2)} \zeta(u-(4p+1))\zeta(1-u) x^u du \\
&= \frac{2^{4p+2}\pi^{4p+2}}{x^{4p+2}} \frac{1}{2\pi i} \int_{4p+5/2-i\infty}^{4p+5/2+i\infty} 2^{u-1} \\
&\times \frac{\pi^u}{\sin(\pi(4p+3-u)/2)} \zeta(u-(4p+1))\zeta(1-u)(x/\pi^2/2^2)^u du.
\end{aligned}$$

Now

$$\sin(\pi(4p+3-u)/2) = \sin(\pi(1-u)/2 + \pi(2p+1))$$

$$= -\sin(\pi(1-u)/2) = \sin(\pi(-1-u)/2) = -\sin(\pi(u+1)/2).$$

We have shown that

$$\boxed{S(x;p) = \frac{B_{4p+2}(2\pi)^{4p+2}}{(8p+4)\times x^{4p+2}} + \frac{B_{4p+2}}{8p+4} - \frac{(2\pi)^{4p+2}}{x^{4p+2}} S(4\pi^2/x;p).}$$

In particular we get

$$S(2\pi;p) = \frac{B_{4p+2}}{8p+4}.$$

The sequence in p starting from $p = 1$ is

$$\frac{1}{504}, \frac{1}{264}, 1/24, \frac{43867}{28728}, \frac{77683}{552}, \frac{657931}{24}, \frac{1723168255201}{171864}, \ldots$$

We thus have for $p = 3$ as per request from OP

$$\boxed{\sum_{n\ge 1} \frac{n^{13}}{e^{2\pi n} - 1} = \frac{1}{24}.}$$

This was math.stackexchange.com problem 3557171.

References

[FS96] Philippe Flajolet and Robert Sedgewick. The average case analysis of algorithms : Mellin transform asymptotics. Technical Report RR-2956, INRIA, Rocquencourt, 1996.

[Szp01] Wojciech Szpankowski. Mellin transform and its applications. In *Average Case Analysis of Algorithms on Sequences*, pages 398–441. John Wiley and Sons, Inc., 2001.

[Vep06] Linas Vepstas. On Plouffe's Ramanujan identities. *arXiv Mathematics e-prints*, September 2006.