Counting toroidal factorizations

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The following is inspired by OEIS A335078 and OEIS A335079 where cyclic factorizations are counted. Here we use integer factorization and Dirichlet series.

Counting weak ordered factorizations

A weak ordered factorization of n into m factors is a sequence of m positive integers that multiply to n and order is taken into account (that's why we use the term sequence). We use the label weak to indicate that one is a valid factor. Call the number of such factorizations for a given n, $F_m(n)$. This function is multiplicative. We see by inspection that its Dirichlet series is $\zeta(s)^m$. Computing it for a prime power p^k it results that it is the number of non-negative solutions of

$$b_1 + \cdots + b_m = k$$

which by stars-and-bars gives $\binom{k+m-1}{m-1}$. Hence with $v_p(n)$ which gives the maximal power of p that divides n we have

$$F_m(n) = \prod_p \binom{v_p(n)+m-1}{m-1}.$$

Counting toroidal factorizations

We count factorizations where the factor set does not include one (that is why weak is not used here). Now suppose we have a torus obtained by attaching opposite sides of a $q \times r$ rectangle with q rows and r columns to each other which yields $m = qr$ slots. Given a factorization of n into m factors we distribute these in some way into the slots. The rotations (symmetries) of the torus are those that are induced by rotating i.e. shifting the source rectangle horizontally or vertically some number of times, with wrap-around. We consider two factorizations to be equivalent when there is a combination of rotations that transform one into the other. Suppose the torus is traced by rotating a circle C around some axis A in the same plane as C, then the two types of symmetries rotate about A or about the circle traced by the center of C as it rotates, which does not preserve metrics. (Here we are concerned with the topology only.) We introduce $G_{q,r}(n)$ which counts these inequivalent factorizations of n into m factors. Recall the cycle index of the cyclic group

$$Z(C_m) = \frac{1}{m}\sum_{d|m} \varphi(d) a_d^{m/d}.$$

Next compute the cycle index of the action of the rotations of the torus on the slots, which can be done by inspection and yields

$$Z(C_{q,r}) = \frac{1}{qr}\sum_{d|q}\sum_{f|r} \varphi(d)\varphi(f) a_{\mathrm{lcm}(d,f)}^{\gcd(d,f)\times q/d\times r/f}.$$

For simplicity call the exponent E, a function on four variables. Now apply the Polya Enumeration Theorem on Dirichlet series with the factor set being $\zeta(s)-1$ to get

$$G_{q,r}(n) = [n^{-s}]\frac{1}{qr}\sum_{d|q}\sum_{f|r} \varphi(d)\varphi(f)(\zeta(\mathrm{lcm}(d,f)s)-1)^E.$$

Expanding the binomial we find

$$[n^{-s}]\frac{1}{qr}\sum_{d|q}\sum_{f|r} \varphi(d)\varphi(f)\sum_{k=0}^{E}\binom{E}{k}(-1)^{E-k}\zeta(\mathrm{lcm}(d,f)s)^k.$$

Observe that the zeta function term counts weak ordered factorizations into k factors where all the factors are

$\mathrm{lcm}(d,f)$th powers through its Dirichlet series. Therefore only those divisors d, f count where n is a $\mathrm{lcm}(d,f)$th power e.g. n must be a sixth power when $\mathrm{lcm}(d,f) = 6$. Write this with an Iverson bracket

$$[n^{-s}]\frac{1}{qr}\sum_{d|q}\sum_{f|r}[[\ n^{1/\mathrm{lcm}(d,f)} = \lfloor n^{1/\mathrm{lcm}(d,f)}\rfloor\ ]]\ \varphi(d)\varphi(f)\sum_{k=0}^{E}\binom{E}{k}(-1)^{E-k}\zeta(\mathrm{lcm}(d,f)s)^k.$$

Actually doing the coefficient extraction we find that

$$[n^{-s}]\zeta(\mathrm{lcm}(d,f)s)^k = [(n^{1/\mathrm{lcm}(d,f)})^{-s}]\zeta(s)^k = F_k(n^{1/\mathrm{lcm}(d,f)}).$$

We thus have the closed form (here we have omitted $k = 0$ as it makes no contribution)

$$G_{q,r}(n) = \frac{1}{qr}\sum_{d|q}\sum_{f|r}[[\ n^{1/\mathrm{lcm}(d,f)} = \lfloor n^{1/\mathrm{lcm}(d,f)}\rfloor\ ]]$$
$$\times\ \varphi(d)\varphi(f)\sum_{k=1}^{E}\binom{E}{k}(-1)^{E-k}F_k(n^{1/\mathrm{lcm}(d,f)}).$$

In practice the test for n being an exact power would be done without using floating point arithmetic as we are factorizing n anyway for the computation of F so we may just check that all the exponents are multiples of $\mathrm{lcm}(d,f)$.

Note that when $\gcd(q,r) = 1$ we can do additional simplification,

$$G_{q,r}(n) = \frac{1}{m}\sum_{d|q}\sum_{f|r}[[\ n^{1/d/f} = \lfloor n^{1/d/f}\rfloor\ ]]$$
$$\times\ \varphi(df)\sum_{k=1}^{m/d/f}\binom{m/d/f}{k}(-1)^{m/d/f-k}F_k(n^{1/d/f})$$
$$= \frac{1}{m}\sum_{d|m}[[\ n^{1/d} = \lfloor n^{1/d}\rfloor\ ]]\varphi(d)\sum_{k=1}^{m/d}\binom{m/d}{k}(-1)^{m/d-k}F_k(n^{1/d}).$$

This no longer depends on the particular choice of q and r but only on their product m. Here we recognize OEIS A335078 and OEIS A335079 which count cyclic factorizations. And indeed if the choice of q as a divisor of m is free as long as $\gcd(q, m/q) = 1$ we may take $q = 1$ and $r = m$ and obtain a torus that is isomorphic to an ordinary cycle of m nodes. This will furnish the value of $G_{q,r}(n)$ through $G_{1,m}(n)$.

Invariance of the cycle index, a conjecture

We can prove something even stronger, namely that $G_{q,r}(n)$ only depends on the product $m = qr$ and $\gcd(q,r)$ which we denote γ. This conjecture should be proved by showing that the cycle indices are the same, before even entering into Dirichlet series. It was verified for quite a number of pairs (q,r) and the corresponding values of $Z(C_{q,r})$ where equality was observed with $Z(C_{q',r'})$ when $\gcd(q,r) = \gcd(q',r')$ and $qr = q'r'$.

Counting factorizations into m factors

A natural statistic to consider is for n given what is the number of toroidal factorizations on some torus with m slots. This is given by

$$Q_m(n) = \sum_{d|m} G_{d,m/d}(n).$$

For example consider $Q_{12}(2^2 3^{12}) = Q_{12}(2125764)$ which gives the constituent terms

$$444, 450, 444, 444, 450, 444$$

that sum to 2676. We observe that when $\mu(m) \neq 0$ (products of primes) we obtain

$$Q_m(n) = d(m) G_{1,m}(n).$$

The corresponding sequence lists $Q_m(n)$ for m up to $\Omega(n)$ for increasing n and starts like this:

$$0, 1, 1, 1, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 2, 1, 1, 4, 2, 1,\\ 1, 2, 1, 2, 1, 4, 2, 3, 1, 1, 4, 2, 1, 1, 4, 2,\\ 1, 2, 1, 2, 1, 1, 6, 6, 3, 1, 2, \ldots$$

It is suggested the reader simplify $Q_m(n)$ supposing that the conjecture holds.

Counting factorizations into any number of factors

Here we add the contributions from all factorizations as follows:

$$Q(n) = \sum_{m=1}^{\Omega(n)} Q_m(n)$$

to get the sequence

$$0, 1, 1, 3, 1, 3, 1, 5, 3, 3, 1, 7, 1, 3, 3, 10, 1, 7, 1, 7,\\ 3, 3, 1, 16, 3, 3, 5, 7, 1, 11, 1, 14, 3, 3, 3, 24, 1, 3,\\ 3, 16, 1, 11, 1, 7, 7, 3, 1, 35, 3, 7, \ldots$$